Pre-reading for Lecture 5b: Rotational Dynamics

- The main thing we’ll be doing in lecture today is “promoting” scalar rotational quantities like angular velocity and torque so they become vector quantities.

The basic vector rotational quantity is the angular velocity vector \( \vec{\omega} \). Its direction tells you the axis of rotation, and its magnitude tells the speed of rotation. The direction of rotation (clockwise or counterclockwise) is determined using the “right-hand rule,” as shown in the figure at right.

Once we’ve decided to make angular velocity a vector, lots of other rotational quantities must also become vectors. We can summarize the vector rotational quantities, along with their linear analogues:

- **Velocity** \( \vec{v} \) ⇔ **Angular velocity** \( \vec{\omega} \)
- **Acceleration** \( \vec{a} = \frac{d\vec{v}}{dt} \) ⇔ **Angular acceleration** \( \vec{\alpha} = \frac{d\vec{\omega}}{dt} \)
- **Force** \( \vec{F} \) ⇔ **Torque** \( \vec{\tau} \)
- **Mass** \( m \) ⇔ **Moment of inertia** \( I \)
- **(linear) Momentum** \( \vec{p} = m\vec{v} \) ⇔ **Angular momentum** \( \vec{L} = I\vec{\omega} \)
- **Newton’s 2nd**: \( \sum \vec{F} = \frac{d\vec{p}}{dt} \) ⇔ **Newton’s 2nd (rotation)**: \( \sum \vec{\tau} = \frac{d\vec{L}}{dt} \)

To define the vector torque, we’ll need to identify the vector force \( \vec{F} \) applied to an object as well as the radial vector \( \vec{R} \) that points from the axis of rotation to the point of application of that force. The vector torque is defined as:

\[
\vec{\tau} = \vec{R} \times \vec{F}
\]

This definition requires a new kind of vector product, called the **cross product**. There are many ways of multiplying two vectors: you’ve already seen the **dot product**, and there are others like the **tensor product** that we won’t need to use in PS2. The cross product takes two vectors and multiplies them to yield a vector. We’ll show you how to define and use the cross product in class; it’s difficult to understand it just by reading about it. Based on your previous lecture, though, you should expect that the magnitude of the torque must be given by \( \tau = RF \sin \theta \) where \( \theta \) is the angle between the vectors \( \vec{R} \) and \( \vec{F} \).

As suggested by the table above, we’ll also define the angular momentum of a rotating system by analogy with the linear case: angular momentum \( \vec{L} = I\vec{\omega} \). Then, just as we have Newton’s Second Law for linear motion, relating the net force on an object with the rate of change of the object’s linear momentum with time:
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\[ \sum \vec{F} = \frac{d\vec{p}}{dt} \]

we have Newton’s Second Law for rotational motion, relating the net torque on an object with the rate of change of the object’s angular momentum with time:

\[ \sum \vec{\tau} = \frac{d\vec{L}}{dt} \]

For an isolated object (with no net torque), the angular momentum must be conserved. Some of the strangeness of rotational motion arises from the fact that both the angular velocity and the moment of inertia can change—so when you change the angular momentum, it’s not simply due to a change in the angular velocity. In contrast, with linear momentum \( \vec{p} = m\vec{v} \) we usually have objects whose mass doesn’t change.

Finally, when a rotating object does not have a fixed axis, you can apply a torque that is perpendicular to the angular momentum vector. In this case, the torque does not change the speed of rotation, but instead changes the direction of the rotation axis, in a phenomenon known as precession. Precession may seem like an exotic phenomenon only of interest to physicists, but it turns out that understanding precession will be crucial for our understanding of how MRI works. We’ll learn about MRI in detail is PS3 next semester once we’ve learned about magnetism.

In addition, it turns out that precession is responsible for the curious fact that the astrological “signs” that were defined thousands of years ago are no longer correct! If you were born in late March or early April, you probably think of yourself as an Aries. Around 600 BCE, that meant that when you were born the Sun appeared in the constellation Aries. But today, thanks to precession, the sun was probably in Pisces during your birth! See: http://www.livescience.com/4667-astrological-sign.html.

• Learning objectives: After this lecture, you will be able to…

1. Identify the angular velocity vector \( \vec{\omega} \) for an object rotating around some arbitrary axis, using the right-hand rule to assign the correct direction of the vector.

2. Define and use the **cross product**, which multiplies two vectors to yield a third vector.

3. Determine the **vector torque** resulting from a force applied at some point on an object using the definition of the cross product and the right-hand rule.

4. Write Newton’s Second Law for rotational motion, and compare this law with the analogous law for linear motion.

5. Use the conservation of angular momentum to predict the behavior of an isolated system if its moment of inertia changes or if some aspect of its angular velocity changes.

6. Predict the direction of precession if a torque is applied to a rotating system in such a way that the torque is perpendicular to the angular momentum vector.

7. Calculate the rate of precession for a rotating system with a perpendicular torque.
Am I getting it?

- You apply a force to a door (the hinge is indicated with a dot). In each case, indicate whether the resulting torque is **positive**, **negative**, or **zero**.

1. ![Diagram 1](image1)

2. ![Diagram 2](image2)

3. ![Diagram 3](image3)

4. ![Diagram 4](image4)
The angular velocity vector

- Suppose there is an object somewhere that is rotating around an axis. What information would we need to give a complete characterization of its rotation?

- We can combine all three of these quantities to form an axial vector known as \( \vec{\omega} \), the angular velocity vector. How does it work?

- What is the angular velocity vector in each of the following examples?
Cross Product and Right-Hand Rule

- We will need a new kind of vector product, called the **cross product**. Consider any two vectors $\vec{a}$ and $\vec{b}$. Define a vector $\vec{c}$, the cross product, as follows:

\[
\vec{c} = \vec{a} \times \vec{b}
\]

- The cross product $\vec{c}$:
  - is a **vector**
  - has magnitude $|\vec{c}| = |\vec{a}| |\vec{b}| \sin \theta$
  - has direction perpendicular to both $\vec{a}$ and $\vec{b}$
  - has orientation given by the **right hand rule**
  - is NOT commutative: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

- To find the direction of the cross product $\vec{c} = \vec{a} \times \vec{b}$, use your **right hand**:
  - Take your **right hand**
  - Hold out your **hand** straight, pointing in the direction of $\vec{a}$.
  - Bend your **fingers** so your fingertips point in the direction of $\vec{b}$.
  - Stick out your **thumb**, which will point in the direction of $\vec{c} = \vec{a} \times \vec{b}$.

- Now we can write a concise expression for the **torque**:

\[
\vec{\tau} = \vec{R} \times \vec{F}
\]
Activity 1: Torque and Angular Momentum

1. You’re applying a force to turn a door (the dot represents the hinge). Find the direction of the vector torque in each of the following situations:

We’ve seen two equivalent versions of Newton’s Second Law for linear motion.

\[ \sum \vec{F} = \frac{d\vec{p}}{dt} \quad \sum \vec{F} = m\vec{a} \]

Let’s find the analogous equations for rotational motion.

2. We defined (linear) momentum \( \vec{p} = m\vec{v} \). Using the rotational analogies we developed in the previous lecture, how should we define angular momentum \( \vec{L} \)?

\[ \vec{L} = \]

3. Here are two versions of Newton’s Second Law for rotational motion. Are they equivalent? Why or why not?

\[ \sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad \sum \vec{\tau} = I\vec{\alpha} \]

- Bonus! Suggest an experiment that could determine which of these versions is correct.
Activity 2: Conservation of Angular Momentum

- Consider an isolated system, for which the net torque is zero.

1. What must be true about the angular momentum of that system? Will it change over time? How can you express this fact mathematically? (Hint: recall that this is the rotational analogue of linear momentum \( \vec{p} \).)

2. Let’s think about the magnitude of the angular momentum: \( L = I\omega \). Note that \( I \) is always positive, \( L \) is a magnitude, and \( \omega \) is a magnitude, so all quantities in this equation are positive. Think of \( \omega \) as the angular speed—a positive quantity.

   Based on what you stated in question (1) above, if the net torque on a system is zero, and the moment of inertia \( I \) decreases, what must happen to the angular speed \( \omega \)?

- Now consider the vector nature of angular momentum. I’m going to sit at rest on a seat that can rotate freely. In my hand is a bicycle wheel that is spinning with its angular velocity vector straight up, as shown at right.

3. What is the total angular momentum for the system (me plus the wheel?)

4. I’m going to flip the wheel upside down, so that its angular velocity points down. In order for angular momentum to be conserved, I must rotate on the stool. Explain why, and predict what direction I will rotate.
Activity 3: Off-Axis Torques

- I’m going to hang a spinning bicycle wheel from a chain (or string). Consider the pivot to be the point where the bike’s axis attaches to the chain.

1. Which direction does each of the following vectors point? (+x, -x, +y, -y, +z, or -z)
   - The angular velocity vector
   - The angular momentum vector
   - The force of gravity
   - The vector $\mathbf{R}$ from the pivot to the point of application of the force of gravity (relevant for the torque equation)
   - The torque vector
   - The change in the angular momentum vector after a short period of time (hint: think about Newton’s 2nd Law for rotation)

- **Bonus!** Precession arises because $\vec{\tau}$ is perpendicular to the angular momentum $\vec{L}$. What happens in linear motion when $\vec{F}$ is perpendicular to the linear momentum $\vec{p}$?
Further Details about Precession

• Here are some more details about precession. You will learn more about this in the lab!

- For additional practice, do the same analysis we did on the previous page for a spinning top (see image below).

- The angular frequency of precession, \( \omega_{\text{precess}} = \frac{mgR}{I \omega_{\text{spin}}} \)

- Super Bonus! Prove this expression is true. In this example when the torque is provided by gravity, does the rate of precession change if we change the angle of tilt?

- Precession is fundamental to MRI (as you’ll learn next semester). Also, it has interesting implications for astrology: [http://www.livescience.com/4667-astrological-sign.html](http://www.livescience.com/4667-astrological-sign.html)
One-Minute Paper

Your name: _____________________________       TF: _____________________________

Names of your group members:  _________________________________
_________________________________
_________________________________

•  Please tell us any questions that came up for you today **during** lecture. Write “nothing” if no questions(s) came up for you during class.

•  What single topic left you **most confused** after today’s class?

•  Any other comments or reflections on today’s class?