Goal: Count regions in hyperplane arrangements.

Known so far:
- boolean arrangement
- braid arrangement
- general position in $\mathbb{R}^2$

Intersection poset

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$.
Let $L(\mathcal{A})$ be the set of non-empty intersections of $\mathcal{A}$, including $\mathbb{R}^n$. The intersection poset of $\mathcal{A}$ is made of $L(\mathcal{A})$ with the relation $x \leq y$ iff $y \subseteq x$ (reverse inclusion).

Examples

Braid arrangement

$D_3$

Intersections $\leftrightarrow$ Set partitions

By...
Given some intersection $x \in L(2^n)$,

$$\text{rank}(x) = n - \# \text{blocks in the set partition corresponding to } x.$$  

**Boolean arrangement** + arrangements of $1R^n$ with $n$ hyperplanes in general position

$n=3$:

$$\{(0,0,0)\}$$

$H_1, H_2, H_3$ intersecting in general position

**Boolean Lattice**:

Intersections $\longleftrightarrow$ Subsets of $\{0,1\}$

$$\text{Rank}(x) = \# \text{hyperplanes}$$

Arrangements of hyperplanes in general position with more than $n$ hyperplanes:

**Proposition**

- The intersection poset always has a $\hat{0}$ ($1R^n$).
- It has a unique maximal element iff $\mathcal{A}$ is central.
- $L(\mathcal{A})$ is a lattice iff $\mathcal{A}$ is central; it is otherwise a meet semi-lattice (every pair of elements has a meet).
Counting regions

Definition

Choose $H_0 \in A$. Let $A' = A - \{H_0\}$ and $A'' = \{H_0 \cap H \neq \emptyset : H \in A'\}$ (so $A''$ are the non-empty intersections with $H_0$).

We call $(A, A', A'')$ a triple of arrangements with distinguished hyperplane $H_0$.

Example

\[ \text{Diagram showing arrangements $A$, $A'$, and $A''$ with hyperplane $H_0$.} \]

Lemma 1 (Generalized sweep method)

Let $(A, A', A'')$ be a triple of arrangements with distinguished hyperplane $H_0$. Then,

\[ r(A) = r(A') + r(A'') \]

and

\[ b(A) = \begin{cases} b(A') + b(A'') & \text{if } \text{rank}(A) = \text{rank}(A') \\ 0 & \text{if } \text{rank}(A) = \text{rank}(A') + 1 \end{cases} \]
Definition

The characteristic polynomial $\chi_A(t)$ of the arrangement $A$ is defined by

\[ \chi_A(t) = \sum_{x \in L(A)} \mu(x) \cdot t^{\dim(x)}, \]

where $\mu(x)$ is the Möbius function.

Example

The braid arrangement $B_3$ has characteristic polynomial

\[ \chi_{B_3}(t) = t^3 - 3t^2 + 2t \]

Lemma 2 (deletion-restriction)

Let $(A, A', A'')$ be a triple of arrangements with distinguished hyperplane $H_0$. Then

\[ \chi_A(t) = \chi_{A'}(t) - \chi_{A''}(t). \]

Sketch of proof

$L(A')$ contains the items of $L(A)$ that are not above $H_0$, and those above $H_0$ that are also above at least two hyperplanes of $A'$. $L(A'')$ is isomorphic to the subposet of $L(A)$ made of the items above $H_0$ in $L(A)$. 
There fore,

\[ \chi_A(t) = \sum_{x \in \mathcal{L}(A)} \mu(x) t^{\dim(x)} \]

\[ = \sum_{x \in \mathcal{L}(A) \setminus \mathcal{H}_0} \mu(x) t^{\dim(x)} + \sum_{x \in \mathcal{L}(A) \cap \mathcal{H}_0} \mu(x) t^{\dim(x)} \]

\[ = \sum_{x \in \mathcal{L}(A) \setminus \mathcal{H}_0} \mu(x) t^{\dim(x)} + \sum_{x \in \mathcal{L}(A) \cap \mathcal{H}_0} \mu_L(x) t^{\dim(x)} \]

\[ = \sum_{x \in \mathcal{L}(A)} (\mu_L(x) - \mu_L(x)_{\mathcal{H}_0}) t^{\dim(x)} \]

\[ = \chi_{A'}(t) - \chi_{A''}(t) \]

**Example**

\( A = \) Boolean arrangement in \( \mathbb{R}^3 \)

\[ \chi_A(t) = t^3 - 3t^2 + 3t - 1 \]

\[ \chi_{A'}(t) = t^3 - 2t^2 + t \]

\[ \chi_{A''}(t) = t^2 - 2t + 1 \]

We are now ready to prove the main theorem about regions (next class).

**Theorem (Zaslavsky, 1975)**

Let \( A \) be a hyperplane arrangement in \( \mathbb{R}^n \). Then,

\[ r(A) = (-1)^n \chi_A(-1) = \| \chi_A(-1) \| \]

and

\[ b(A) = (-1)^\text{rank}(A) \chi_A(1) = \| \chi_A(1) \| \]

**Reference:** [Sta07], Lectures 1 and 2.