Math 108 - Geometric Combinatorics   Nadia Lafrenière

Intersection poset and counting regions   1/19/2023

Goal: Count regions in hyperplane arrangements

Known so far:
- boolean arrangement
- braid arrangement
- general position in $\mathbb{R}^2$

Intersection poset

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$.

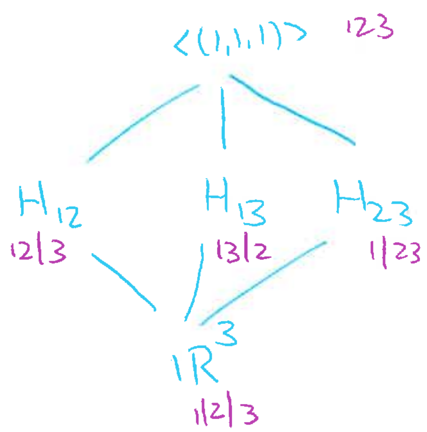
Let $L(\mathcal{A})$ be the set of non-empty intersections of $\mathcal{A}$, including $\mathbb{R}^n$. The intersection poset of $\mathcal{A}$ is made of $L(\mathcal{A})$ with the relation $x \leq y$ iff $y \subseteq x$ (reverse inclusion).

Examples

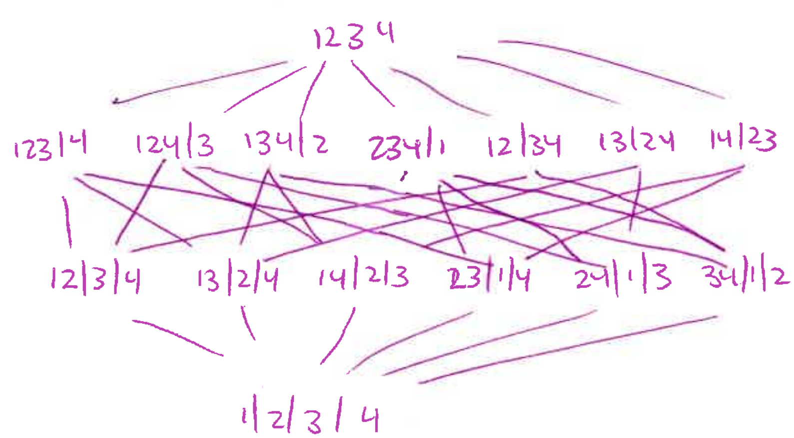
Braid arrangement

Intersections <——> Set partitions

$\mathcal{B}_3$



$\mathcal{B}_4$

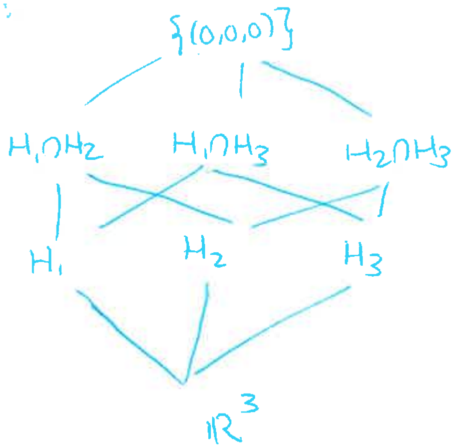


Given some intersection $x \in L(\mathcal{B}_n)$,

$$\text{rank}(x) = n - \#\left\{\begin{array}{l}\text{blocks in the set partition}\\ \text{corresponding to } x\end{array}\right\}.$$

Boolean arrangement + arrangements of $\mathbb{R}^n$ with n hyperplanes in general position

$n=3$:

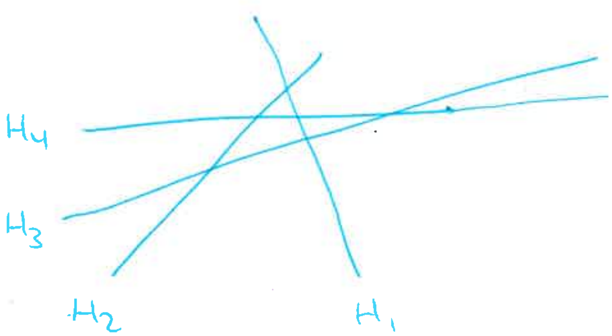


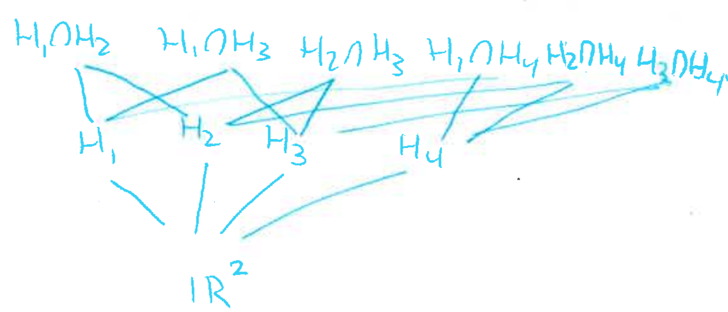
Boolean Lattice!

Intersections $\longleftrightarrow$ Subsets of $[n]$

Rank (x) = # hyperplanes

Arrangements of hyperplanes in general position with more than n hyperplanes:





Proposition

- The intersection poset always has a $\hat{0}$ ($\mathbb{R}^n$).
- It has a unique maximal element iff $\mathcal{A}$ is central
- $L(\mathcal{A})$ is a lattice iff $\mathcal{A}$ is central; it is otherwise a meet semi-lattice (every pair of elements has a meet).

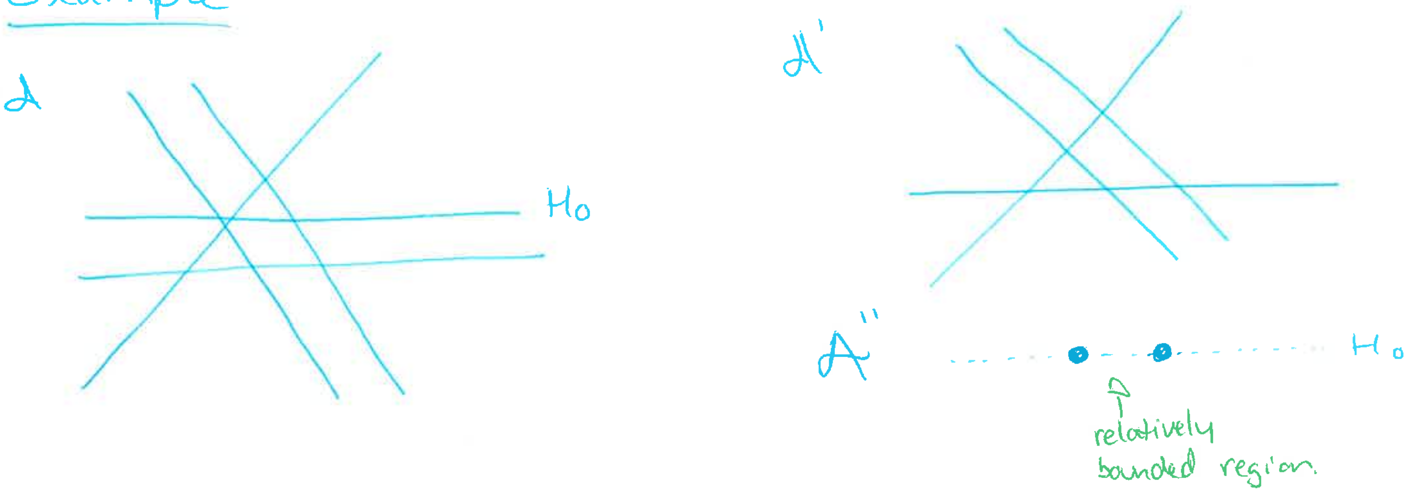
Counting regions

Definition

Choose $H_0 \in \mathcal{A}$.
Let $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \{H_0 \cap H \neq \emptyset : H \in \mathcal{A}'\}$ (so $\mathcal{A}''$ are the non-empty intersections with H_0).

We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a <u>triple</u> of arrangements with <u>distinguished hyperplane</u> H_0.

Example



Lemma 1 (Generalized sweep method)

Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane H_0. Then,

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$$

and

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') \\ 0 & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1 \end{cases}$$

Definition

The characteristic polynomial $\chi_{\mathcal{A}}(t)$ of the arrangement $\mathcal{A}$ is defined by

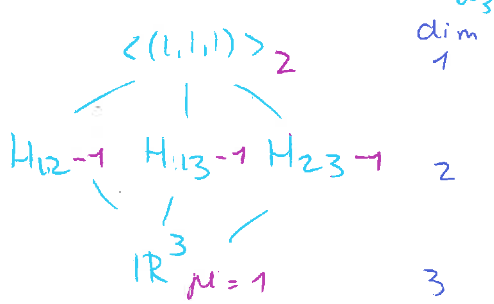
$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot t^{\dim(x)},$$

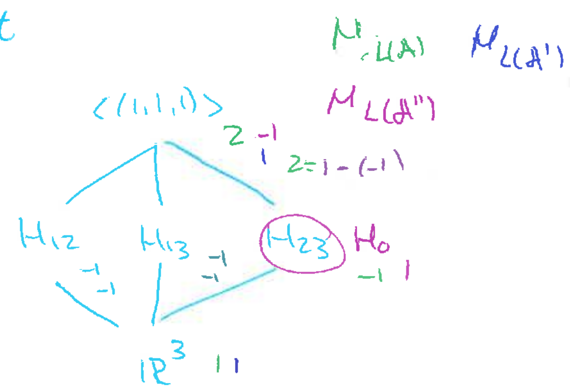
where $\mu(x)$ is the Möbius function.

Example

The braid arrangement $\mathcal{B}_3$ has characteristic polynomial

$$\chi_{\mathcal{B}_3}(t) = t^3 - 3t^2 + 2t$$





Lemma 2 (deletion-restriction)

Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane H_0. Then

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

Sketch of proof

$L(\mathcal{A}')$ contains the items of $L(\mathcal{A})$ that are not above H_0, and those above H_0 that are also above at least two hyperplanes of $\mathcal{A}'$. $L(\mathcal{A}'')$ is isomorphic to the subposet of $L(\mathcal{A})$ made of the items above H_0 in $L(\mathcal{A})$.

Therefore,

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x)\, t^{\dim(x)}$$

$$= \sum_{\substack{x \in L(\mathcal{A}) \\ x \not\geq H_0}} \mu(x)\, t^{\dim(x)} + \sum_{\substack{x \in L(\mathcal{A}) \\ x \geq H_0}} \mu(x)\, t^{\dim(x)}$$

$$= \sum_{\substack{x \in L(\mathcal{A}') \\ x \not\geq H_0}} \mu_{L(\mathcal{A}')}(x)\, t^{\dim(x)} + \sum_{\substack{x \in L(\mathcal{A}'') \\ x \geq H_0}} \underbrace{\mu_{L(\mathcal{A})}(x)}_{\mu_{L(\mathcal{A}')}(x) - \mu_{L(\mathcal{A}'')}(x)}\, t^{\dim(x)}$$

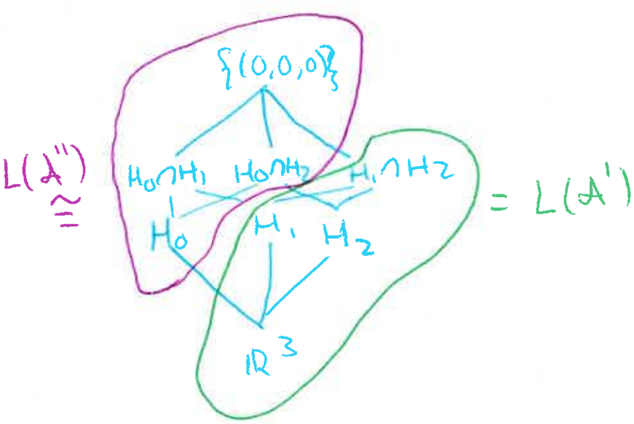
$\mu_{L(\mathcal{A}')}(x) - \mu_{L(\mathcal{A}'')}(x)$, since $\mu_{L(\mathcal{A})}(H_0) = -1 = -\mu_{L(\mathcal{A}'')}(H_0)$.

$$= \sum_{x \in L(\mathcal{A})} \left(\mu_{L(\mathcal{A}')}(x) - \mu_{L(\mathcal{A}'')}(x)\right) t^{\dim(x)}$$

$$= \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

Example

$\mathcal{A}$ = Boolean arrangement in $\mathbb{R}^3$.



$$\chi_{\mathcal{A}}(t) = t^3 - 3t^2 + 3t - 1$$

$$\chi_{\mathcal{A}'}(t) = t^3 - 2t^2 + t$$

$$\chi_{\mathcal{A}''}(t) = t^2 - 2t + 1.$$

We are now ready to prove the main theorem about regions (next class).

Theorem (Zaslavsky, 1975)

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then,

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1) = |\chi_{\mathcal{A}}(-1)|$$

and

$$b(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1) = |\chi_{\mathcal{A}}(1)|.$$

Reference: [Sta07], Lectures 1 and 2.