Math 108- Geometric Combinatorics

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Brower Fixed point Theorem, Sperner's Lemma

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Similarly to what we did with the Borsuk-Ulam Theorem and Tucker's lemma, we present a classical theorem in topology and its discrete analogue.

Theorem (Brouwer, 1912)

Every continuous function $f: B^n \to B^n$ on the ball has a fixed point, i.e., there exists x such that $x=f(x)$.

We will give two proofs of Brouwer's Theorem: one using the Borsuk-Ulam Theorem, and one algorithmic.

Lemma (Corollary of the Borsuk-Ulam Theorem)

There is no continuous function $f: B^n \to S^{n-1}$ that is antipodal on the boundary of B^n.

Proof

Assume the opposite, and let $f: B^n \to S^{n-1}$ be antipodal on the boundary and continuous.

Furthermore, let $\pi: S^n \to B^n$ be such that $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$.

Consider the upper hemisphere $\mathcal{U}$ of S^n to be $x \in S^n$ s.t. $x_{n+1} \geq 0$.

upper hemisphere

let

$$g: S^n \to S^{n-1}$$

$$x \mapsto f\circ\pi(x) \quad \text{if } x \text{ in } \mathcal{U}.$$

$$x \mapsto -g(-x) \quad \text{if } x \text{ not in } \mathcal{U}.$$

Then, g is well-defined, continuous (because it is continuous on each hemisphere and antipodal on the Equator) and antipodal on $S^n \to S^{n-1}$, contradicting the Borsuk-Ulam Theorem. Therefore, there exists no such function f.

$\boxtimes$

We can prove that the above lemma also implies the Borsuk-Ulam theorem, making it equivalent to it, but it is not needed here.

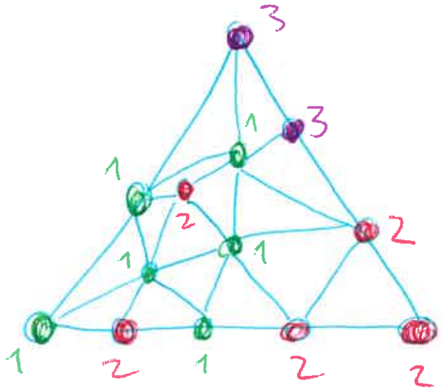
Proof (of Brouwer's Theorem)

By contradiction. Let $f: B^n \to B^n$ be continuous with no fixed point. Let $g: B^n \to S^{n-1}$ be the intersection of S^{n-1} with the half-line from $f(x)$ to x (which is well-defined, since f has no fixed points). Then, $g(x) = x$ for all $x \in S^{n-1} = \partial B_n$, which means that $g(-x) = -x = -g(x)$ on the boundary, and g is continuous, contradicting the above lemma. ∎

Sperner's Lemma

We now consider triangulations of simplices, in which we can add vertices not only on the interior, but also on the boundary.

Example $d=2$.



Definition

Let T be a triangulation of the simplex $\mathrm{conv}(\{e_1, \dots, e_n\})$. Then, a Sperner coloring of T is a labeling of $\mathrm{vert}(T)$ satisfying $\lambda(v) \in \{i \in [n+1] : v_i \neq 0\}$.

This means that, in a triangle, every corner has one option for color, other vertices on the edges have two, and there are three options on the interior.

An n-dimensional simplex is called fully labeled if its n+1 vertices all have different labels.

Theorem (Sperner's Lemma, 1928)

Consider a Sperner coloring of a triangulation of the n-dimensional simplex. Then, there exists a fully labeled simplex in the triangulation. More precisely, the number of fully labeled n-simplices is odd.

Theorem

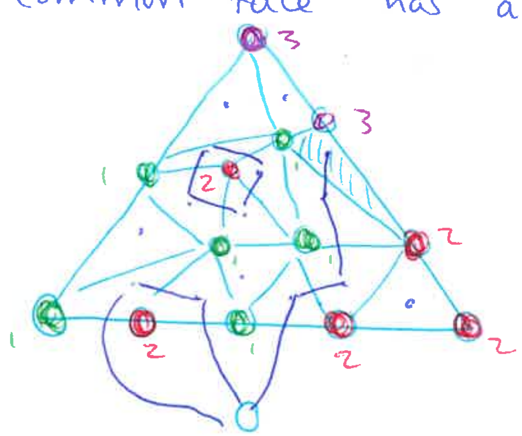
Sperner's Lemma and the Brouwer fixed point theorem are equivalent.

We first prove Sperner's Lemma combinatorially. Then, we prove the equivalence of the theorems.

Proof

The construction uses a graph.

- Vertices: Simplices of the triangulation + 1 "outside" vertex
- Edges: Two adjacent simplices share an edge if their common face has all the labels [n].



Claims: - Vertices have degree 0, 1 and 2 (except for the outside vertex), and have degree 1 exactly when the simplex is fully labeled.

- The outside vertex has odd degree (can be proved by induction).

Therefore, we want to show that the number of vertices of odd degree, besides the outside vertex, is odd. By the Handshake lemma (from graph theory), the total number of vertices of odd degree is even (because the sum of the degrees is even), and therefore the number of fully labeled simplices is odd, proving Sperner's Lemma. ☒

Proof of the equivalence

We give the proof of Brouwer => Sperner, and give an idea for the other direction.

B => S Consider a triangulation of the n-simplex and a Sperner labeling λ of it. Define the function

$$f: v \mapsto e_{\lambda(v)+1 \bmod (n+1)},$$

which "rotates the boundary by 1/n of a turn". Expand f affinely. Therefore, f has no fixed point on the boundary.

Notice that a point $f(x)$ is on the interior of the simplex if and only if x is inside a fully labeled simplex for λ.

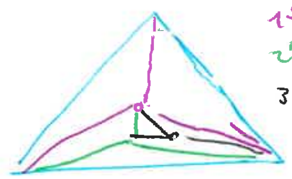
We must show that the image of f has at least one point on the interior of the simplex. Since f has no fixed point on the boundary, Brouwer's Lemma* implies that it has a fixed point x on the inside, so $f(x)$ is also on the interior. Hence,

x is in a fully labeled simplex, and such a simplex exists.* (Note that Brouwers lemma applies to the simplex, because it is homeomorphic to the ball)

S

Ideas for Sperner => Brouwer.

Let f be a continuous function on the simplex whose corners are $\{(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)\}$.
We create a distinguished point using barycentric divisions of the simplex:



At each step, we define a Sperner coloring as

$$\lambda(v) = \min \{ i \mid f(v)_i - v_i < 0 \},$$

and by Sperner's lemma, each subdivision has a fully labeled simplex.
At each division, we only focus on the fully labeled simplex, and divide it again. This process has a limit x. Then,

$$f(x)_i - x_i \le 0 \quad \text{for all } i.$$

Moreover, because both x and $f(x)$ are in the simplex,
$\sum_i x_i = \sum_i f(x)_i = 1.$

Hence, $f(x)_i - x_i = 0$ for all i, and x is a fixed point. ∎

References: [Lon13], §1
[Matousek], §2