<u>Last time</u>: Given two rep^s V, V' of G and their characters $\chi, \chi' : G \to \mathbb{C}$ $(\chi(g) = tr(g : V \to V))$,
$$\dim Hom_G(V, V') = H(\chi, \chi') = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi'(g) \text{ Hermitian inner product.}$$

Combining with Schur's lemma:

- characters of irred. reps of G are <u>orthonormal</u> for H: $H(\chi_i, \chi_j) = \delta_{ij}$.
  in particular, # irred. reps. $\leq$ # conjugacy classes
- in the decomposition of a rep. W into irreducibles $W \simeq \oplus V_i^{\oplus a_i}$, the multiplicities $a_i = H(\chi_{V_i}, \chi_W)$, and $H(\chi_W, \chi_W) = \sum a_i^2$.
- the dimensions of the irreducible rep's satisfy $|G| = \sum (\dim V_i)^2$.

<u>Ex</u>: S_4

	1 e	6 (12)	8 (123)	6 (1234)	3 $(12)(34)$	
U	1	1	1	1	1	trivial
U'	1	-1	1	-1	1	alternating
V	3	1	0	-1	-1	standard
V'	3	-1	0	1	-1	$V' = V \otimes U'$
W	2	0	-1	0	2	found using $\sum \dim^2 = 24$ and orthogonality.

(then interpreted as: $S_4 \underset{\text{quotient by } \mathbb{Z}/2 \times \mathbb{Z}/2}{\twoheadrightarrow} S_3 \underset{\text{standard rep. of } S_3}{\longrightarrow} GL(W)$).

* The other option to construct W is to look at $V \otimes V$: $\chi_{V \otimes V} = \chi_V^2 = (9, 1, 0, 1, 1)$
We have $H(\chi_U, \chi_{V\otimes V}) = 1$, $H(\chi_{U'}, \chi_{V\otimes V}) = 0$, $H(\chi_V, \chi_{V\otimes V}) = \frac{1}{24}(27 + 6 - 6 - 3) = 1$,
$H(\chi_{V'}, \chi_{V\otimes V}) = \frac{1}{24}(27 - 6 + 6 - 3) = 1$, so $V \otimes V$ contains $U \oplus V \oplus V'$ (dim. 7)
and this leaves us one copy of the missing irreducible W. So: $V \otimes V = U \oplus V \oplus V' \oplus W$
(and we can find χ_W by subtracting the others from $\chi_{V\otimes V}$).

<u>Ex</u>: A_4 alternating subgroup of S_4. This has 4 conjugacy classes: $\{e\}$ 1 element

(3-cycles are one conj class in S_4 but split in A_4, see lecture 23) $\begin{cases} (123) & 4 \\ (132) & 4 \end{cases}$

$(12)(34)$ 3

→ We can start by restricting to A_4 the irred. rep's of S_4 - some become isomorphic (eg the alternating rep. U' has elements of A_4 acting by $(-1)^{6} = 1$ so $\simeq$ trivial), others might become reducible. This is feasible but tricky (largely W's fault).

→ Or we can go at it directly! We know there's at most 4 irred. reps, of $\sum \dim^2 = 12$, including the trivial rep^n of dim 1 ⇒ the only option is $12 = 3^2+1^2+1^2+1^2$.

The three 1-dim^l representations correspond to $\mathrm{Hom}(A_4, \mathbb{C}^*) \ni id$ (trivial rep) and two other elements...

Observe $H = \{id\} \cup \{(ij)(kl)\}$ normal subgroup,

$A_4/H \simeq \mathbb{Z}/3$, so this gives the answer: $\mathrm{Hom}(A_4, \mathbb{C}^*) \simeq \widehat{\mathbb{Z}/3} = \{m \mapsto e^{2\pi i m k/3}\}$.

Concretely, let $\lambda = e^{2\pi i/3}$, then the rank 1 rep's are:

(Note: $W_{1_{A_4}} \simeq U' \oplus U''$)

	e	(123)	(132)	$(12)(34)$
U	1	1	1	1
U'	1	λ	λ^2	1
U''	1	λ^2	λ	1
V	3	0	0	-1

U', U'': $(ij)(kl) \in H$ act by id

and the last one by orthogonality is: V. This is the restr. to A_4 of the standard rep. of S_4!

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* Last time we said but didn't prove: characters of irreducible rep's are actually an <u>orthonormal basis</u> (for H) of the space of class functions $G \to \mathbb{C}$.

The proof uses a more general averaging/projection formula.

Last time we saw: $\varphi_V = \frac{1}{|G|}\sum_{g\in G} g : V \to V$ projection onto the invariant subspace V^G (= trivial summands in V)

<u>Prop:</u> Given any class function $\alpha : G \to \mathbb{C}$ and any rep^n V of G, let $\varphi_{\alpha,V} = \frac{1}{|G|}\sum_{g\in G} \alpha(g)\, g : V \to V$.

Then $\varphi_{\alpha,V} : V \to V$ is G-linear (equivariant).

<u>Proof:</u> $\varphi_{\alpha,V}(hv) = \frac{1}{|G|}\sum_{g\in G} \alpha(g)\, g h v$

$= \frac{1}{|G|}\sum_{g'\in G} \alpha(hg'h^{-1})(hg'h^{-1}) hv = \frac{1}{|G|}\sum_{g'\in G} \alpha(g')\, hg'v$

relabel sum: $g = hg'h^{-1}$.

$= h\left(\frac{1}{|G|}\sum_{g'\in G} \alpha(g')g'v\right) = h.\varphi_{\alpha,V}(v)$. □

→ <u>Thm:</u> The characters of the irreducible rep's of G form an orthonormal basis (for H) of the space of class functions $G \to \mathbb{C}$, and # irred reps = # conjugacy classes.

<u>Proof:</u> To show the characters $\chi_1, \dots, \chi_m$ of the irred. reps span all class functions, it suffices to show: $H(\bar{\alpha}, \chi_i) = 0\ \forall i \Rightarrow \alpha = 0$.

Given any class function α and an irreducible rep. V, $\varphi_{\alpha,V} : V \to V$ as above.

Then by Schur's lemma, $\varphi_{\alpha,V} = \lambda . id_V$, where $\lambda = \frac{1}{n} tr(\varphi_{\alpha,V})$, $n = \dim V$.

So: $\lambda = \frac{1}{n} tr(\varphi_{\alpha,V}) = \frac{1}{n} \frac{1}{|G|} \sum_{g\in G} \alpha(g)\, tr(g) = \frac{1}{n}\frac{1}{|G|} \sum \alpha(g) \chi_V(g) = \frac{1}{n} H(\bar{\alpha}, \chi_V)$.

<u>So</u>: if $H(\bar{\alpha}, \chi_{V_i}) = 0$ $\forall V_i$ irreducible, then $\varphi_{\alpha, V_i} = 0$ $\forall V_i$, hence by considering direct sums, $\varphi_{\alpha, V} = 0$ for all rep's of G, in particular for the regular representation R of G (permutation rep. for left-mult on G).

So: for the regular representation, $\varphi_{\alpha,R}(e_1) = \frac{1}{|G|} \sum_{g\in G} \alpha(g)\, e_g = 0$.

Since the e_g are linearly indep^t, this implies $\alpha(g) = 0$ $\forall g \in G$, ie. $\alpha = 0$. □

Along the way, we found:

For V_i, V_j irreducible, look at $\varphi_{\alpha, V_j} : V_j \to V_j$ for $\alpha = \overline{\chi_{V_i}}$: then

$$\varphi_{\alpha, V_j} = \lambda . id_{V_j} \quad \text{where } \lambda = \frac{1}{\dim V_j} tr(\varphi_{\alpha, V_j}) = \frac{1}{\dim V_j} H(\chi_{V_i}, \chi_{V_j}) = \begin{cases} \frac{1}{\dim V_j} & i = j \\ 0 & i \neq j \end{cases}$$

$\Rightarrow$ <u>Prop</u>: if V is any rep. of G and $V = \bigoplus V_i^{\oplus a_i}$ its decomposition into irreducibles, then $\varphi_{V_i} = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)}\, g$, $V \longrightarrow V$ is the projection onto the summand $V_i^{\oplus a_i}$ (ie. identity on that summand, 0 on others).

(The case of the trivial rep.^{n} = our previous projection formula for V^G).

<u>The representation ring of G</u>:

Fix a group G and consider the set of (finite dim., /$\mathbb{C}$) representations of G up to isomorphism. There are two operations $\oplus$ and $\otimes$ which are commutative, associative, and distributive $(U \oplus V) \otimes W = (U \otimes W) \oplus (V \otimes W)$. So this is a ring?.. almost! We're missing additive inverses. We'll just add them!

Let $\hat{R} = \{ \sum_{\text{finite}} a_i [V_i] \,/\, a_i \in \mathbb{Z},\ V_i \text{ reps of } G \}$ formal linear combinations with integer coefficients of rep's of G

and consider the additive subgroup generated by all $[V] + [W] - [V \oplus W]$.

Let $R(G)$ = the quotient of $\hat{R}$ by this subgroup.

(so, in $R(G)$, $[V] + [W] = [V \oplus W]$, but we can subtract rep's!).

$(R(G), \oplus, \otimes)$ is now a ring – the <u>representation ring of G</u>

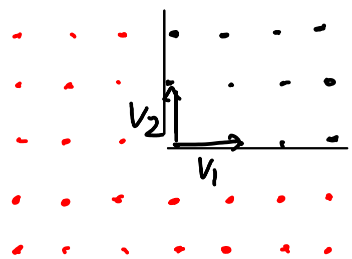
extend these operations to formal sums / differences of rep's by linearity!

As a set, $R(G) = \left\{ \sum_{i=1}^{k} a_i V_i \;\middle|\; a_i \in \mathbb{Z} \right\}$ where V_i = the irreducible representations of G

(complete reducibility + uniqueness of decomposition into irreps.)

ie. $(R(G), +)$ is a free abelian group ($\simeq \mathbb{Z}^k$, $k = \#$irreducibles).

General elements ($a_i \in \mathbb{Z}$) are called "virtual representations"; actual rep's, ie elements st. $a_i \geq 0$ $\forall i$, form a cone inside it. (ie. subset closed under addition).



the cone of rep's inside $R(G)$.

virtual rep's.

Next: the character, $V \mapsto \chi_V$, can be extended by linearity to a map $R(G) \longrightarrow \mathbb{C}_{class}(G)$. This is a ring homomorphism!

class functions

$(\chi_{U \oplus V} = \chi_U + \chi_V, \ \chi_{U \otimes V} = \chi_U \chi_V)$

The image of this map = "virtual characters" $\left(= \left\{ \sum a_i \chi_{V_i}, \ a_i \in \mathbb{Z} \right\} \right)$.

Passing to complex linear combinations instead of integer ones, our results about irred. characters forming a basis say:

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\simeq} \mathbb{C}_{class}(G) \quad \text{is an isomorphism}$$

$$\sum_{i=1}^{k} a_i [V_i] \longmapsto \chi_{\sum a_i V_i} = \sum a_i \chi_{V_i}$$

$(a_i \in \mathbb{C}$ now$)$

(tensor product of (free) $\mathbb{Z}$-modules, works same as for vector spaces).

- There are theorems of Artin and Brauer that describe the lattice of virtual characters $\Lambda = \left\{ \sum a_i \chi_{V_i}, \ a_i \in \mathbb{Z} \right\}$ inside $\mathbb{C}_{class}(G)$. We'll see those after Thanksgiving.

Next time: we'll look at rep's of S_5 and A_5, for extra practice with characters + to motivate discussion of restriction & induction of representations (rep's of $G \longleftrightarrow$ rep's of subgroups of G).