2. Principle of Inclusion–Exclusion

Let S be a set of size k. Then, the number of subsets of S of size k is given by:

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \]

Proof. Let \( A \) be a set of subsets of \([n]\) of size \( k \)

Let \( A_1 \) be a set of subsets of \([n]\) of size \( k-1 \)

Let \( A_2 \) be a set of subsets of \([n]\) of size \( k+1 \)

Then, the number of subsets of \([n]\) of size \( k \) is:

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \]

Binomial Theorem

\[ (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \]

Proof. By induction

Base case: \( n = 0 \)

Inductive step: Assume \( (1+x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i \)

Then, \( (1+x)^{m+1} = (1+x)(1+x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i \cdot \sum_{j=0}^{m} \binom{m}{j} x^j \)

By the Binomial Theorem,

\[ \sum_{i=0}^{m} \binom{m}{i} x^i \cdot \sum_{j=0}^{m} \binom{m}{j} x^j = \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m}{i} \binom{m}{j} x^{i+j} \]

By the binomial theorem,

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

Principle of Inclusion–Exclusion (PIE)

For all \( A, B \),

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

For all \( A, B \) disjoint,

\[ |A \cup B| = |A| + |B| \]

For all \( A, B \) not disjoint,

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

Example: How many numbers from 1 to 100 are divisible by 2 or 5?

\[ |A| = 50 \]

\[ |B| = 20 \]

\[ |A \cup B| = \frac{|A| + |B| - |A \cap B|}{2} \]

By PIE,

\[ |A \cup B| = \frac{50 + 20 - 10}{2} = 30 \]

By PIE,

\[ |A \cap B| = \frac{|A| + |B| - |A \cup B|}{2} \]

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By PIE,

\[ |A \cup B| = \frac{50 + 20 - 10}{2} = 30 \]

By PIE,

\[ |A \cap B| = \frac{|A| + |B| - |A \cup B|}{2} \]

By PIE,
\[ n \leq y \leq z \]

\[ \emptyset = \{ x \mid 1 \leq x \leq 100, \text{x is divisible by 2 and 5} \} \]

\[ (y, \in \mathbb{Z}) \]

\[ \leq 0 \]

\[ \text{x is divisible by 10} \]