We will begin the course by reviewing the theory of Bose & Fermi liquids before going on to discuss strongly correlated systems.

**Weakly interacting Bose fluid**

Ideal Bose gas at $T=0$: $H_0 = \sum_i \frac{p_i^2}{2m}$

Ground state: All bosons occupy $\mathbf{p} = 0$ state

- Bose condensate

$$1^g \varphi_g \propto \left( a^+_b = 0 \right)^N 1_0 >.$$  

($N = \# \text{ of particles}$).

(I will assume familiarity with 2nd quantization; see eg pages 39-50 of Altland, Simons, or many other texts).

Excitations: Remove 1 boson from condensate and let it have momentum $\mathbf{p} \rightarrow \text{quadratic dispersion}$. 

Note: We have fixed the total # of particles, and the ground and excited state we constructed have the same # of particles. It will soon be convenient however to consider a Hilbert space where we allow the total # of particles to be arbitrary (a "grand canonical" point of view). This is sometimes called a Fock space. The operators $a^\dagger$ or $a$ really live in this bigger space with varying particle #.

Weakly interacting Bose system:

$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_i \sum_j V(\vec{x}_i - \vec{x}_j)$$

Assume $V$ is repulsive, 2 short-ranged

(i.e. $V(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$ "sufficiently fast")

In 2nd quantized form

$$H = \sum_k \frac{\hbar^2 k^2}{2m} a^\dagger_k a_k + \frac{1}{2} \int d^3x \ a^\dagger(x) a^\dagger(x') V(x-x') a(x) a(x')$$
In $k$-space interaction term becomes

$$H_{int} = \frac{1}{2} \sum_{k, k'} V(q) a_{k+q}^+ a_k^+ a_k a_{k+q}$$

with $V(q) = \int d^d x \ e^{-i \frac{x}{2} q} V(x)$.

If $V(q)$ is "weak" we may still expect that a macroscopic # of particles are in the $f=0$ state, i.e., in the condensate.

Call this $N_0$; $N_0 = \langle a_{k=0}^+ a_{k=0} \rangle$.

Expect $N_0 \lesssim N$, more precisely $\frac{N-N_0}{N} \ll 1$ if $V(q)$ is weak.

$$\langle a_0^+ a_0 \rangle = N_0 \Rightarrow \text{matrix elements of } a_0 \leq a_0^+ \text{ are } o(\sqrt{N_0})$$

Also $[a_0, a_0^+] = 1 \Rightarrow \text{RHS} \ll \text{typical matrix element of } a_0 \text{ or } a_0^+$.

When $N \gg 1$, can ignore commutators & treat $a_0$ and $a_0^+$ as complex #s ($o(\sqrt{N_0})$)

(i.e. meanphase is $\sqrt{N_0}$)
Approximate treatment of interaction: Retain only those terms from $H_{int}$ that involve interaction of uncondensed particles with condensate.

Illustrate with simple model $V(q) = V$, i.e.

$$V(x-x') = V S^0(x-x')$$

("contact interaction")

$$H_{int} = \frac{V}{2L^d} \sum \frac{a^+ \, a \, a^+ \, a}{k_1 \, k_2 \, k_1 + q \, k_2 - q}$$

$$= \frac{V}{2L^d} \sum \frac{N_0^2}{L^d} + \frac{V}{2L^d} \sum \frac{a^+ \, a \, a^+ \, a}{k \, k \, -k \, -k}$$

$$+ \frac{1}{2} \left( a^+ \, a \, a^+ \, a \right)$$

1st term: Self-interaction energy of condensate

2nd term: "Hartree-Fock" energy of uncondensed particles interacting with condensate.

3rd term: Creation/annihilation of particles from condensate.

Write $N = N_0 + \sum_{k \neq 0} a^+ \, a$.
\[ V \frac{N_0^2}{2L^d} = \frac{V}{2L^d} \left( N - \sum_{k \neq 0} a_k^+ a_k \right)^2 \]

\[ = \frac{VnN}{2} - Vn \sum_{k \neq 0} a_k^+ a_k \]

\((n = N/L^d = \text{total density})\)

\[ H = \frac{VnN}{2} - Vn \sum_{k \neq 0} a_k^+ a_k + Vn \sum_{k \neq 0} \left[ a_k^+ a_k + a_k^+ a_{-k} \right. \]

\[ + \frac{1}{2} \left( a_{-k}^+ a_k + a_k^+ a_{-k}^+ \right) \]

\[ + \sum_{k \neq 0} \frac{k^2}{2m} a_k^+ a_k \]

\[ = \frac{VnN}{2} + \sum_{k \neq 0} \left( \frac{k^2}{2m} + Vn \right) a_k^+ a_k \]

\[ + Vn \left( a_{-k}^+ a_k + a_k^+ a_{-k}^+ \right) \]

\(H\) diagonalize consider the general Hamiltonian

\[ \hat{H} = \sum_k \left( t_k a_k^+ a_k + \delta_k \left( c_{-k}^+ a_k + a_k^+ c_{-k}^+ \right) \right) \]

Here \( t_k = \epsilon_k + Vn \), \( \delta_k = Vn/2 \) and \( \epsilon_k = \frac{k^2}{2m} \).

Diagonalize by writing

\[ a_k = u_k \xi_k + v_k \xi_{-k}^+ \]

\[ a_k^+ = u_k \xi_k^+ + v_k \xi_{-k} \]
I choose \( u_k, v_k \) so that
\[
\begin{align*}
\{v_k, v'_{k'}\} &= 0, \\
\{v_k, v^+_{k'}\} &= \delta_{kk'}
\end{align*}
\]
and
\[
\Pi = \text{const.} + \sum_k E_k v_k v^+_k
\]

Doing this gives (see HW)
\[
E_k = \sqrt{(E_k + \nu n)^2 - V^2 n^2}
\]
\[
= \sqrt{E_k (E_k + 2\nu n)}
\]

The \( E_k \) are the excitation energies of "normal modes" or small oscillations of the Bose condensate.

For small \( k \), \( E_k = \hbar c |k| \)

with \( c = \hbar \sqrt{\frac{\nu n}{m}} \)

while for large \( k \),
\[
E_k \approx \frac{\hbar^2 k^2}{2m}
\]

Thus dispersion of excited modes becomes linear at low \( |k| \) (as opposed to quadratic in the non-interacting theory).

We can calculate the depletion of the condensate due to interactions.
\[
\frac{N - N_0}{N_0} = - \frac{1}{2} \sum_{k \neq 0} \langle jd | a_k^+ a_k | gd \rangle \\
= \frac{L^d}{N^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \left( \frac{k^2}{2m} + mV \frac{1}{E_k} \right)
\]

In d=3 this can be evaluated to give
\[
\frac{N - N_0}{N} = \frac{1}{3 a^2 n} \left( \frac{mc}{\hbar} \right)^3 \sim (mV)^{3/2} n^{1/2}
\]

The linear dispersive mode at small \(|k|\) corresponds physically to a sound wave - oscillations of the density of the fluid (Elaborate later).

Comment: In this approximation ("Bogolyubov" theory) the particle number is not conserved.

Physically this is due to presence of condensate which acts like an "infinite" source/sink of particles.

This is related to the phenomenon of broken symmetry to which we turn next.

**Broken symmetry in the Bose fluid**

Consider the expectation value \( \langle a^+(x) a(x') \rangle \)
\[
= \frac{L^d}{N} \sum_{k,k'} e^{i k x - i k' x} e^{i k - i k'} \langle a_k^+ a_k \rangle
\]
Broken Symmetry in the Bose fluid

In the theory we developed so far, \( a^\dagger, a \) were treated classically.

In the full quantum treatment this means that in the ground state \( \langle gd | a^\dagger | gd \rangle \neq 0 \), \( \langle gd | a | gd \rangle \neq 0 \)

( & both \#s have magnitude \( \sim \sqrt{N_0} \)).

Consider \( a(x) = \frac{1}{\sqrt{L^d}} \sum_k e^{i \mathbf{k} \cdot \mathbf{x}} a_k \)

\[ \langle a(x) \rangle = \frac{1}{\sqrt{L^d}} \langle a_k \rangle \] (\( k \) to modes have \( \langle a_k \rangle = 0 \) by translation invariance).

\[ \therefore \left| \langle a(x) \rangle \right| = \frac{1}{\sqrt{L^d}} \left| \langle a_k \rangle \right| = \sqrt{\frac{N_0}{L^d}} \sim o(1) \]

Thus we have \( \langle a(x) \rangle = \phi_0 \)

\( \langle a^\dagger(x) \rangle = \phi_0^* \)

with \( \phi_0 \) a complex \# of magnitude as \( N \to \infty \).

Clearly then \( |gd\rangle \) is a coherent superposition of states with different total \#s \( N \).
In position space, the 2nd quantized Hamiltonian is

$$H = \int d^d x \ a^+(x) \left( -\frac{\hbar^2 \nabla^2}{2m} - \mu \right) a(x) + \frac{1}{2} \int d^d x \ int d^d x' \left( a^+(x) a(x') \right) V(x, x') \left( a^+(x') a(x') \right)$$

(including a chemical potential term \( \mu \)).

\(H\) has no matrix elements between states with different particle \#; i.e., total particle \# \( \int d^d x \ a^+(x) a(x) \) is conserved by \(H\).

There is a corresponding symmetry of the Hamiltonian:

If we let \(a(x) \rightarrow e^{i\alpha} a(x)\) (i.e., therefore \(a^+(x) \rightarrow e^{-i\alpha} a^+(x)\)) with \(\alpha\) independent of \(x\), then \(H\) is unchanged.

(Note: The set of all transformations \(a(x) \rightarrow e^{i\alpha} a(x)\) form a group, the \(U(1)\) group; thus \(H\) is said to have a global \(U(1)\) symmetry.)

Though this is a symmetry of \(H\), a state with \(\langle a \rangle = \psi_0 \neq 0\) is not invariant under this symmetry.
If $\alpha \to e^{i\theta} \alpha$, then $\psi \to e^{i\theta} \psi$.

Thus the ground state does not share the symmetry of the Hamiltonian, i.e., we have a broken symmetry.

$\langle \alpha \rangle = \psi_0$ is the "order parameter" that quantifies the extent of broken symmetry.

In the symmetry broken phase, the order parameter must be included as an additional thermodynamic variable in a microscopic description of the state.

The magnitude of the order parameter is fixed by microscopic energetics.

The phase of the order parameter is however arbitrary.

(If it could be chosen for instance by a weak coupling of the Bose system we are interested in to another much larger Bose system that also has a "stronger" order parameter).

Changing the magnitude of the order parameter costs energy.

But the phase can be changed without any energy cost.
"Hydrodynamic" theory of the Bose fluid

We now abandon the microscopic treatment and develop a macroscopic description of the low energy, long wavelength physics of the Bose fluid.

The starting point is the assumption that the ground state breaks the global U(1) symmetry.

(Note: We no longer restrict ourselves necessarily to weak interactions; at any interaction strength, so long as the symmetry is broken, our treatment will be valid.)

In the giant state, the order parameter is uniform.

We expect that low energy excited states are obtained by smooth long wavelength distortions of the phase and of the order parameter.

In other words, if we let \( |a(x)\rangle = \psi(x) e^{i\Theta(x)} \)

so long as \( \Theta(x) \) varies slowly on microscopic length scales we expect a state of low energy.
Apart from the order parameter phase $\Theta(x)$, we also expect
the particle density $\rho(x)$ to have slow dynamics
like it is conserved, a fluctuation in $\rho(x)$ cannot
just decay — it can only relax by particle flow out of
the region.

Thus, the "hydrodynamic" variables necessary to describe the
long length scale/long time physics of the Bose fluid
are $\Theta(x)$ and $\rho(x)$.

We first show that these are canonically conjugate.

Start with \[ [a(x), a^+(x')] = \delta(x-x'). \]

(If you prefer to start with \[ [a_k, a^+_k] = \delta_{kk'}, \]

then using \[ a(x) = \frac{1}{\sqrt{L^3}} \sum_k e^{-\frac{i}{L^3} k \cdot x} a_k, \]

can easily
derive the real space commutator.)

For states that are smooth long-distance deformations
of the ground state, we can write
\[ a(x) = \sqrt{\rho + \delta \rho(x)} e^{i\Theta(x)} \]

$\delta$ = mean density $\rho$ $\delta \rho$ is a "small" fluctuation, i.e. in
relevant portion of Hilbert space, matrix elements of $\delta \rho(x)$
will have small absolute values compared to \( S_0 \).

\[
\alpha^\dagger(x) \alpha(x) = \left( e^{-i \theta(x)} \sqrt{S_0} \right) \left( \sqrt{S_0} \ e^{i \theta(x)} \right)
\]

\[
\approx S(x).
\]

Now \( [\alpha(x), \alpha^\dagger(x')] = S^{(d)}(x-x') \)

\[
\Rightarrow [\alpha(x), \alpha^\dagger(x) \alpha(x')] = \alpha(x) S^{(d)}(x-x')
\]

\[
\Rightarrow \left[ \sqrt{S_0+S_0} \ e^{i \theta(x)}, S(x') \right] = \sqrt{S_0+S_0} \ e^{i \theta(x)} S^{(d)}(x-x')
\]

Now approximate \( \sqrt{S_0+S_0} \approx \sqrt{S_0} \) in front of \( e^{i \theta(x)} \)

\[
\Rightarrow \left[ e^{i \theta(x)}, S(x') \right] = e^{i \theta(x)} S^{(d)}(x-x').
\]

Thus for small fluctuations about ground state \( \alpha(x) \) will deviate only slightly from its constant value in the ground state.

\[
\Rightarrow \text{can then expand in powers of } \theta \text{ and equate leading non-vanishing terms to get}
\]

\[
[\theta(x), S(x')] = -i S^{(d)}(x-x').
\]

(More generally, note that \( S(x) = i \frac{\delta}{\delta \phi(x)} \) satisfies \[
[\ e^{i \phi(x)}, S(x') \] = e^{i \phi(x)} S(x-x').\]
Now consider a generic Hamiltonian for the low energy long distance physics of the Bose fluid. We expect
\[
\hat{H} = \hat{H} \left[ \hat{S}(x), \hat{\theta}(x) \right]
\]
\[
= \int d^d x \; \mathcal{H} \left[ \hat{S}(x), \hat{\theta}(x) \right]
\]
\[
\mathcal{H} = \text{Hamiltonian density and is a function of the fields } S(x), \theta(x) \text{ and their derivatives}.
\]

For small fluctuations about ground state, we expand \( \mathcal{H}[S, \theta] \) in powers of \( S^2, \theta \) and their gradients.

A key restriction is that \( \mathcal{H} \) cannot depend on the absolute value of \( \theta \) itself (as the orientation of the order parameter plays no role in setting the energy).

\( \mathcal{H} \) can only depend on \( \nabla \theta \).

To quadratic order in \( S^2 \geq \nabla \theta \), we have (on symmetry grounds)
\[
\mathcal{H} = \frac{(S^2)^2}{2K} + \frac{S}{2} (\nabla \theta)^2 + \ldots
\]

Stability requires \( K > 0, S > 0 \).
Let us interpret $\tilde{y}$ (later we will interpret $\gamma$).

Note that $\frac{S H}{S \phi(x)} = +\mu(x, t) = +\left(\text{local chemical potential}\right)$

from the explicit expression for $H$, it follows that

$$\mu(x, t) = \frac{SS(x, t)}{K}$$

Thus $K = \frac{dp}{dp} = \text{compressibility of the fluid}$

(change in density for unit change of chemical potential)

$\tilde{y}$ is known as the "phase stiffness" - it measures the energy cost to twisting the phase of the order parameter.

Hydrodynamic equ of motion: we have

$$\left[\Theta(x), \phi(x')\right] = -i\tilde{s}^{(d)}(x-x')$$

and

$$H = \int d^dx \left[ \frac{(\nabla \phi)^2}{2K} + \frac{\epsilon_2}{2} (\nabla \Theta)^2 \right]$$

As in usual

$$\partial \Theta / \partial t = \frac{i}{\hbar} [\Theta, H]$$

or equivalently

$$\partial \Theta / \partial t = + \left(\frac{S \nabla \phi}{S \phi(x)}\right) = +\mu(x, t) = \frac{\delta\phi}{\delta t}$$
Thus the rate of change of phase is determined by the local chemical potential - this is an important result (it is the basis of the ac Josephson effect in superconductors).

The other eqn of motion is

\[ \frac{\partial S}{\partial t} = -\langle \frac{\mathcal{H}}{\hbar} \rangle + \frac{\partial^2 \phi}{\partial \phi^2} \]

Note that I also satisfy the continuity equation

\[ \frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{j} \]

It follows that we can identify \( \mathbf{j} = \) particle current

\[ = -\frac{\phi}{\hbar} \nabla \phi \]

Combining the 2 eqns we see that

\[ \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial \phi^2} \]

\[ \frac{\partial^2 S}{\partial t^2} = \frac{8}{\hbar} \nabla^2 S \]

Thus the linear dispersing mode we found in the microscopic calculation can be understood as a slow fluctuation of the order parameter phase (or equivalently as a density fluctuation).

It is sometimes called zero-sound, \( \phi \) is an example of a "Goldstone" mode associated with this broken symmetry.
Vortices. Next we describe an interesting class of excitations which cost huge energy but are nevertheless stable.

\( \theta \) is defined as a phase 2-heme \( \theta \equiv \theta + 2\pi m \) with \( m \in \mathbb{Z} \).

Consider \( C \) in 3D a cylindrical sample with a hole in the middle.

\( e^{i\theta} \) must be single valued.

\[ \oint_C \nabla \phi \cdot dl = 2\pi m \]

for every curve \( C \) that encloses the hole.

Configurations with different values of \( m \) are topologically distinct and cannot be smoothly deformed into one another.

Can also consider such configurations directly in the bulk.

- \( \theta \) winds by \( 2\pi m \) on encircling some point \( P \).

For consistency, we also need to have \( |\psi_0(x)| \) vanish as we approach \( P \).

These objects are called vortices.
Energy cost of vortices. If we go to distances $r \gg a = \text{"core size" of vortex}$, local gradients of the phase are small $\theta$ we can use the long wavelength theory to calculate the energy.

We have $\mathcal{E}_0 = \frac{m}{r} \mathcal{E}_0$

$\Rightarrow E_{(2d)} = \int d^3x \frac{1}{2} \left( \nabla \theta \right)^2 = \pi L_s^2 m^2 \ln \frac{L}{a}$

(In 2d, $E_{(2d)} = \frac{\pi L_s^2 m^2 \ln \frac{L}{a}}{2L}$).

Despite their huge energy cost, vortices can be produced if we "twist" the boundary conditions at spatial $\infty$ so that $\theta$ winds by $2\pi m$ at the boundary (e.g. in a 2d disc, at the disc boundary).

**Superfluidity.** Return to eqns of motion

$\delta \theta = \mu \Rightarrow \frac{\partial \theta}{\partial t} = \frac{\partial \mu}{\partial \theta}$

Thus if $\theta$ is uniform thru the sample, the system cannot sustain chemical potential differences.

As $\mathbf{j} = -\frac{\hbar}{2} \mathbf{\nabla} \theta$, we have $\frac{\partial \mathbf{j}}{\partial t} = -\frac{\hbar}{2} \mathbf{\nabla} (\nabla \mu)$. 
If there is an applied $\nabla \mu$, the current will increase.

In particular, if $\nabla \mu = 0$, $\partial j / \partial t = 0$, i.e. current flows without degrading (superflow).

Can current ever degrade? Requiring motion of vortex lines.

Consider a 2d section:

```
\begin{tikzpicture}
  \draw (-2,-2) -- (2,-2) -- (2,2) -- (-2,2) -- cycle;
  \draw[->] (-2,-2) -- (-2,2);
  \draw[->] (-2,-2) -- (2,-2);
  \node at (-1,-1) {$\theta_1$};
  \node at (1,1) {$\theta_2$};
\end{tikzpicture}
```

Motion of vortex from left to right

$\Rightarrow \theta_2$ winds by $2\pi$

relative to $\theta_1$.

If $\eta_v$ = rate of vortex motion from left to right

$\dot{\theta}_2 - \dot{\theta}_1 = 2\pi \eta_v = \Delta (\theta_2 - \theta_1)$

(ref. vortex motion produces chem. potential difference)

More insight: Consider a disc with a hole in the middle

Ground state: $\theta$ uniform

No current.
State with a vortex trapped in hole

\[ \oint \mathbf{v} \cdot d\mathbf{l} = 2\pi \]

\( \theta \) is non-uniform circulation

\( \Rightarrow \) there is a non-zero circulating current.

To degrade the current to zero, it is necessary that the trapped vortex escape from the hole.

For a large but finite sample, this takes a long (but finite) time (due either to thermal or quantum energy fluctuations over a large barrier).

- Topological stability of vortex \( \Rightarrow \) persistence of superflow

Variational wave function for the Bose fluid

- Hydrodynamic description above gives description of universal low energy physics of the superfluid state for any interaction strength.

- Bogoliubov theory gives a microscopic treatment that includes non-universal aspects but is valid at weak interaction strength.
Now we describe an approach to study microscopic aspects even away from weak interaction limit using variational wavefunctions which build in some "obvious" physics.

Use a first quantized perspective:

\[ H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i,j} V(x_i - x_j). \]

Try to guess a "good" ground state wavefunction.

Start with the ideal gas \((V = 0)\): what is wavefunction?

\[ |\psi_{\text{gd}}\rangle \propto \left( q^+ \right)^N |0\rangle. \]

\(\Rightarrow\) In 1st quantization

\[ \psi_{\text{gd}}(\vec{x}_1, \ldots, \vec{x}_N) \propto 1 \]

(To be precise \( \psi_{\text{gd}}(\vec{x}_1, \ldots, \vec{x}_N) = \left( \frac{1}{L^{3/2}} \right)^N \)).

Now include the repulsive short-range interaction \(V\).

Expect amplitude to find 2 particles closer than
range of interaction will be suppressed.

But when all particles are far away from each other, except amplitude is similar to that of ideal gas, i.e., is constant.

Other general restrictions:

1. As we are dealing with bosons, $4(C_{x_1}, \ldots, \tilde{x}_N)$ is symmetric under exchange.

2. Ground state wave function is real (from time-reversal) and it does not change sign.

$\Rightarrow$ Can choose $\psi_{\text{bos}}(\tilde{x}_1, \ldots, \tilde{x}_N)$ to be a real symmetric, negative and non-negative function.

A simple guess satisfying all of this is

$$\psi_{\text{bos}}(\tilde{x}_1, \ldots, \tilde{x}_N) \propto \prod_{i,j} f(|\tilde{x}_i - \tilde{x}_j|).$$

where $f(|\tilde{x}|)$ = real, non-negative

and $f(|\tilde{x}| \rightarrow 0) \rightarrow 0$ (for a hard-core potential)
and \(f(1|x| \to \infty) \to \text{const.}\).

In general \(f(1|x|)\) for small \(|x|\) may be taken to be the wavefunction of the 2-particle Schrodinger Eqn.

This is known as the Tashlow wavefunction.

It does well in K.E. at long distances & on interaction energy at short distances.

Choosing some parameterization of \(f(1|x|)\) can minimize variational energy, eg, numerically then Monte Carlo, to obtain reasonable estimates of ground state energy of the interacting Bose fluid.

(Can then calculate, eg, \(\mathbf{K}\), etc.)