**Trees**

**Theorem.** The following are equivalent:

1. \( G \) is connected and acyclic.
2. \( G \) is connected and \( |E| = |V| - 1 \).
3. \( G \) is acyclic and \( |E| = |V| - 1 \).
4. \( G \) is acyclic and \( |E| = |V| - 1 \).
5. \( G \) is acyclic and adding any new edge creates a cycle.

**Proof.**

**1 to 2.** Given a connected graph, if it is acyclic, then \( |E| = |V| - 1 \).

**2 to 3.** Graphs connected by definition.

**3 to 4.** Use strong induction on the number of vertices.

**Base Case:** \( N = 1 \), no paths.

**Inductive Step:** Assume the statement is true for all \( N \) less than \( |V| \).

Pick an arbitrary \( v \in V \), let \( u \in N_v \), \( u \) be a neighbor of \( v \).

Then, \( G' = (V, E \setminus \{(v, u)\}) \) is not disconnected.

Then, \( G' \) has 2 connected components, \( G_1, G_2 \) if \( G' = (V, E') \).

**5 to 1.** Use contradiction, assume \( uv \) in \( G \) is not connected.

Add edge \( (v, u) \) must create a cycle, let the cycle be \( (v, u, v) \).

Since \( G \) is acyclic, this contradicts the assumption that \( uv \) are not connected.

**1 to 5.** Let \( (v, u) \) be a new edge.

Let \( (v, u, v) \) be a path from \( v \) to \( v \).must exist because \( G \) is connected. Then \( (v, u, v) \) is a cycle, a new cycle because \( G \) is acyclic.

**II.** Two descriptors are equivalent, 1-2, 2-1

1. \( G \) is a connected set of objects, \( 1 \).
2. \( G \) is a connected set of objects, \( 2 \).

**III.** \( |E| \geq |V| - 1 \) and \( |E| = |V| - 1 \) if \( |V| \geq 3 \).