Math 108 - Geometric Combinatorics Nadia Lafrenière

Introduction to real hyperplane arrangements

1/18/2023.

Consider the vector space $\mathbb{R}^n$.

A linear hyperplane H is a $(n-1)$-dimensional subspace of $\mathbb{R}^n$, i.e.

$$H = \{ v \in V : \underset{\text{dot product}}{\alpha \cdot v} = 0 \}$$

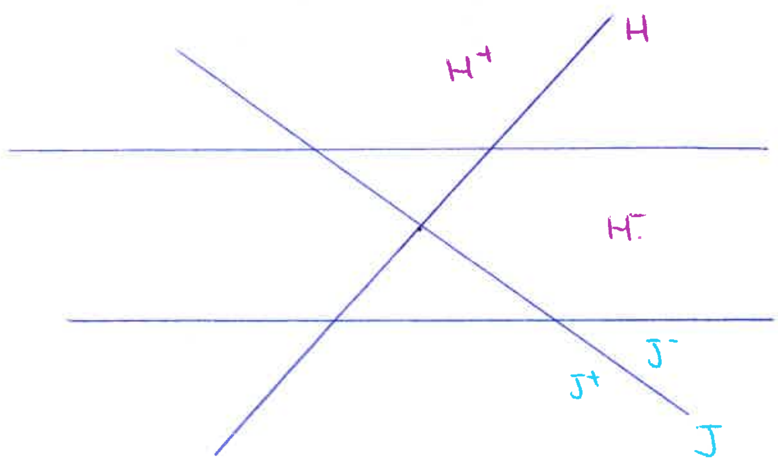
for a fixed vector α (the normal vector).

An affine hyperplane is a translate of a linear hyperplane:

$$J = \{ v \in V : \underbrace{\alpha \cdot v = a}_{\text{fixed}} \}.$$

A hyperplane arrangement is a finite set of hyperplanes in $\mathbb{R}^n$.

A hyperplane H divides $\mathbb{R}^n \setminus H$ into two half-spaces, H^+ and H^-.



Remark: We are equally interested in the hyperplanes as in their complement.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$.

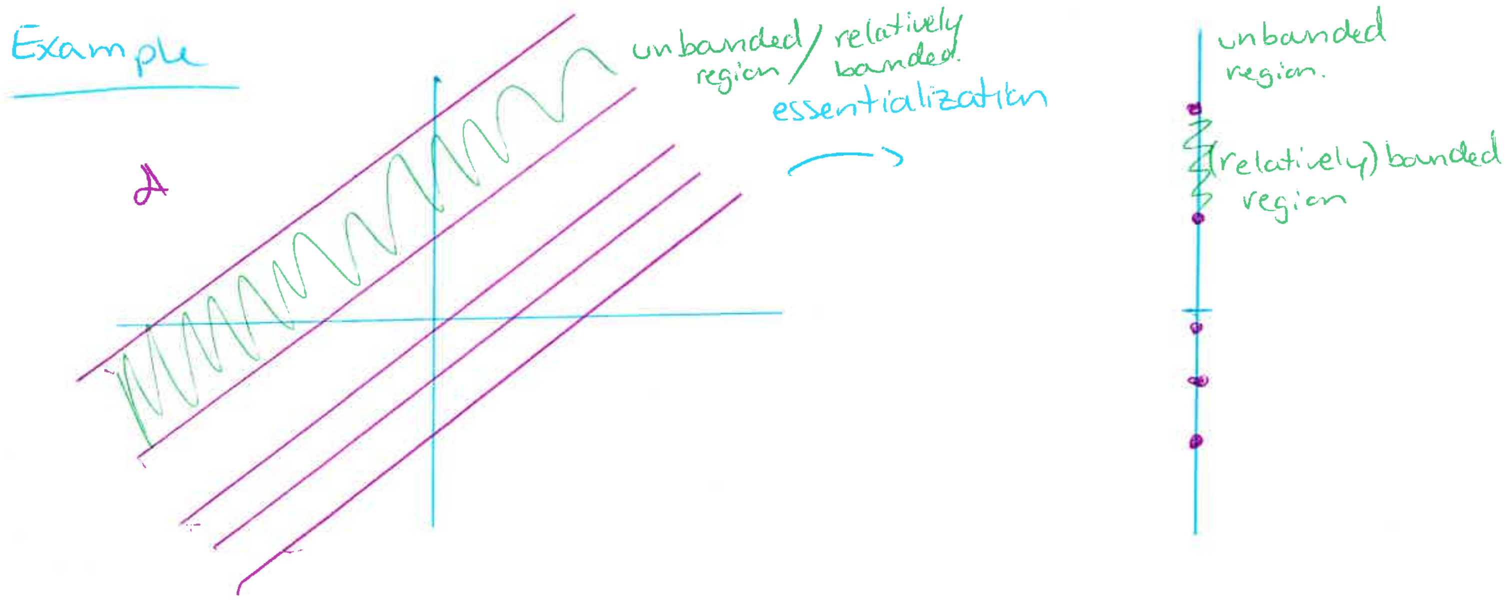
Then,

- the dimension of $\mathcal{A}$ is $\dim(\mathcal{A}) = \dim(\mathbb{R}^n) = n$.
- the rank is the dimension of the space spanned by the normal vectors to the hyperplanes.

A hyperplane arrangement $\mathcal{A}$ is essential if $\dim(\mathcal{A}) = \operatorname{rank}(\mathcal{A})$. ②

If $\mathcal{A}$ is not essential, its essentialization is its projection onto the space spanned by the normal vectors to the hyperplanes.

Example



A region is a connected component of the complement X of the hyperplanes: $X = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$.

We denote $r(\mathcal{A})$ for the number of regions.

We say regions are bounded or unbounded if they are in the geometric sense.

All regions of a non-essential arrangement are unbounded.

If $\mathcal{A}$ is not essential, we say a region is relatively bounded if it is bounded in its essentialization. The number of relatively bounded regions is $b(\mathcal{A})$

Regions are open sets of $\mathbb{R}^n$. The closure of a region R is $\overline{R}$.

A closed half space $\overline{H^+}$ or $\overline{H^-}$ is $H^+ \cup H$ or $H^- \cup H$.

The closure of a region is a finite intersection of closed half spaces.

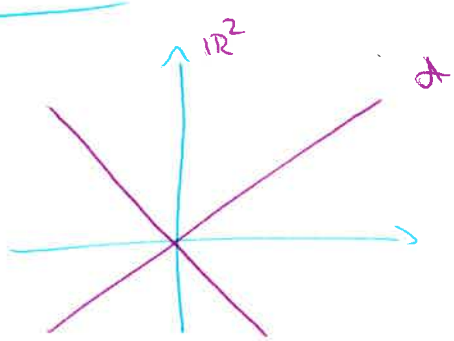
	Closure	Word description of closure
Region R	$\bar{R}$	Convex polyhedra
Bounded region R	$\bar{R}$	Convex polytope
Bounded region in $\mathbb{R}^2$	$\bar{R}$	Convex polygon.

General position

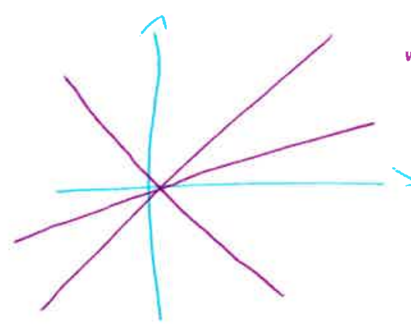
An arrangement $\mathcal{A}$ is in general position if

- $\{H_1, \dots, H_p\} \subseteq \mathcal{A}$, $p \le n \Rightarrow \dim(H_1 \cap \dots \cap H_p) = n-p.$
  $\Leftrightarrow \dim(\langle v_1, \dots, v_p \rangle) = p$ → normal vectors.
- $\{H_1, \dots, H_p\} \subseteq \mathcal{A}$, $p > n \Rightarrow H_1 \cap \dots \cap H_p = \emptyset$

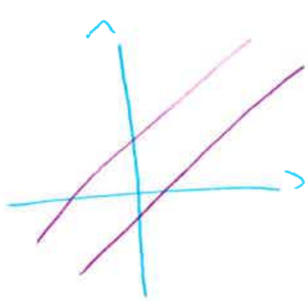
Example



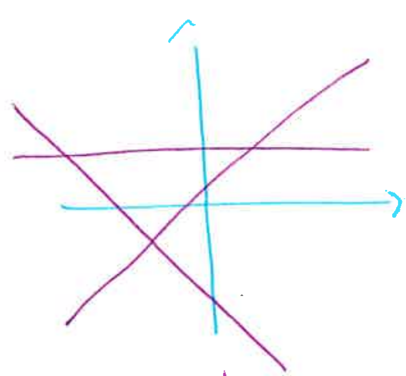
general position



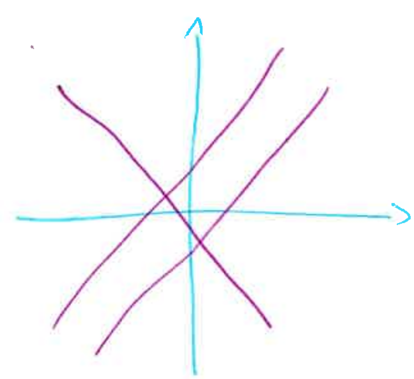
not in general position



not in general position



general position



not in general position.

Proposition (sweep hyperplane method).

Let $\mathcal{A}_m$ be a hyperplane arrangement made of m lines in $\mathbb{R}^2$ in general position. Then, the number of regions is

$$r(\mathcal{A}_m) = \binom{m}{2} + m + 1.$$

Proof (induction on n)

$m=0$: No hyperplane, one region.

Assume this is true for any hyperplane arrangement in general position with m lines.

$\underline{r(\mathcal{A}_{m+1})}$: Choose $H \in \mathcal{A}_{m+1}$. Then $\mathcal{A}_{m+1} \setminus H$ has $\binom{m}{2} + m + 1$ regions, by induction hypothesis

Also, because $\mathcal{A}_{m+1}$ is in general position, no three lines intersect in one point.

For each region that H traverses, it splits it into two.

Therefore,

$$r(\mathcal{A}_{m+1}) = \underbrace{r(\mathcal{A}_{m+1} \setminus H)}_{\binom{m}{2}+m+1} + \#\{\text{regions that } H \text{ traverses}\}$$

The number of regions that H traverses is given by $m+1$, and this is obtained by sweeping H: it intersects each hyperplane once, each time entering a new region, and it is in a region before intersecting the first hyperplane.

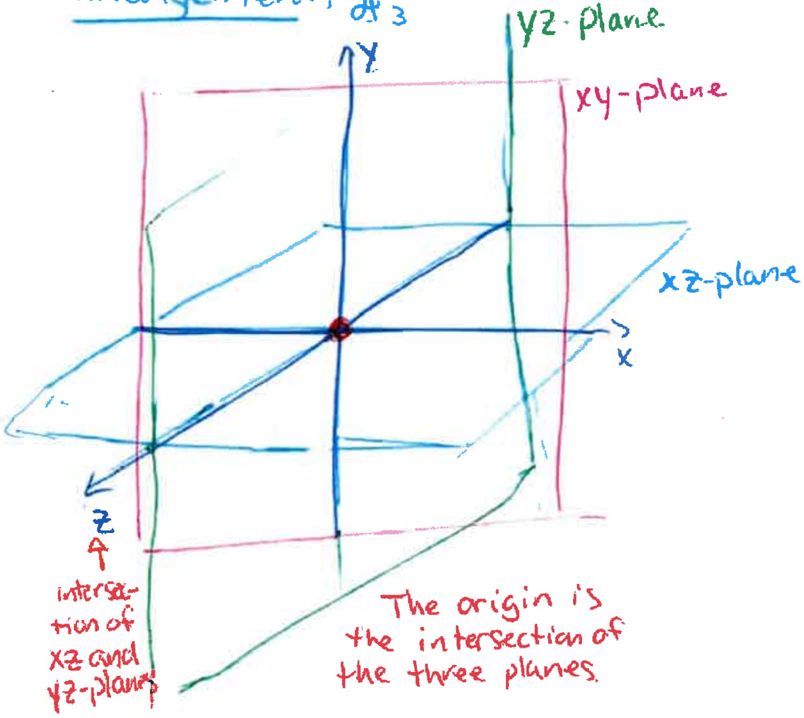
Hence,

$$r(\mathcal{A}_{m+1}) = \binom{m}{2} + m + 1 + m + 1 = \binom{m+1}{2} + (m+1) + 1$$

We have thus proven that the number of regions of $\mathcal{A}_m$ is $\binom{m}{2} + m + 1$.

Interesting hyperplane arrangements

The coordinates hyperplanes, in $\mathbb{R}^3$, form the boolean arrangement, $\mathcal{A}_3$



In $\mathbb{R}^n$:

Dimension n

Rank : n

Essential? Yes

General position? Yes.

Non-empty intersection of all hyperplanes (central)? Yes (origin)

$r(\mathcal{A})$: 2^n

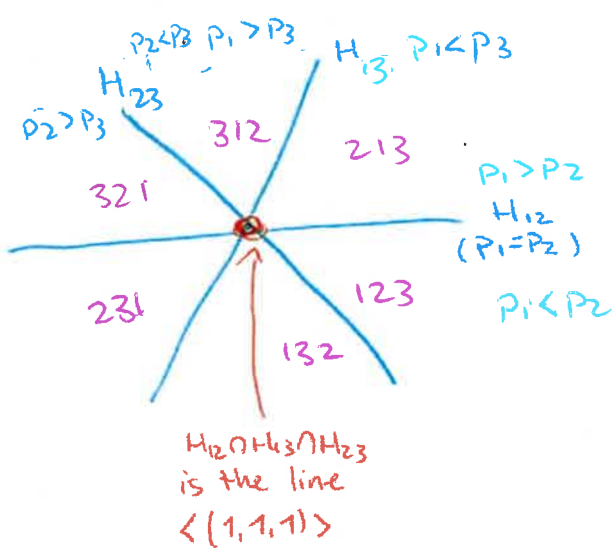
$b(\mathcal{A})$: 0

The braid arrangement

Hyperplanes: $H_{ij} = \{\vec{p} = (p_1, \dots, p_n) \mid p_i = p_j\}$

# of hyperplanes: $\binom{n}{2}$

$\mathcal{B}_3$, the braid arrangement in $\mathbb{R}^3$, admits the following projection onto a 2-dimensional space.



In $\mathbb{R}^n$:

Dimension: n

Rank: $n-1$.

Essential? No

General position? No (central, yet has $>n$ hyperplanes when $n>3$)

Central? Yes ($p_1 = p_2 = \dots = p_n$; line).

$r(\mathcal{A}) = n!$

Argument for $r(\mathcal{A})$:

For each hyperplane H_{ij}, one side corresponds to $p_i < p_j$ and one side to $p_j > p_i$. Therefore, the number of regions corresponds to the number of orderings of $[n]$.

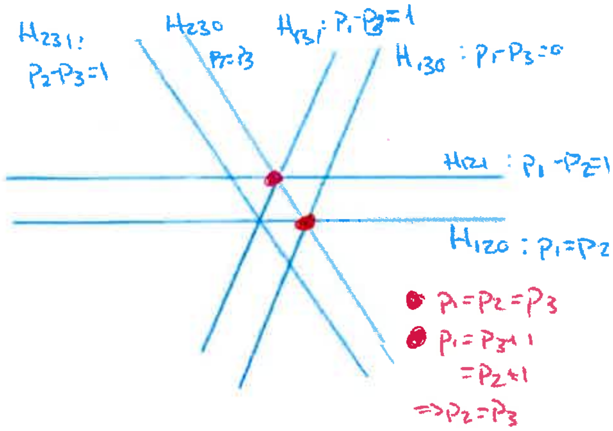
The Shi arrangement

This one is similar to the braid arrangement, but has twice as many hyperplanes:

$H_{ij0} = \{ p \mid p_i = p_j \}$

$H_{ij1} = \{ p \mid p_i - p_j = 1, i < j \}$

The 2-dimensional representation of the 3-dimensional arrangement is



In $\mathbb{R}^n$:

Dimension: n

Rank: $n-1$

Essential? No

General position? No (parallel hyperplanes).

Central? No

Reference: [Sta07] lecture 1