Math 55a Homework 12

Due Wednesday December 2, 2020.

Material covered: Characters and applications (Fulton-Harris §2, §3.1)

0. Please complete the end of semester survey (in Canvas). (Also please participate in the official course evaluation process at the end of the semester! This is really important).

1. Recall that S_4 is the group of rotations of a cube in $\mathbb{R}^3$. Denoting the coordinates by (x, y, z), this gives an action of S_4 on the space V_d of polynomials of degree d (with complex coefficients) in the variables x, y, z. Describe the representations V_1 and V_2, and express them as direct sums of irreducible representations of S_4. Which degree 2 polynomials $f(x, y, z)$ are invariant under S_4?

2. View again S_4 as the group of rotations of a cube in $\mathbb{R}^3$.

(a) Let V be the (8-dimensional) permutation representation associated to the action of S_4 on the set of vertices of the cube. Express V as a direct sum of irreducible representations of S_4.

(b) Do the same for the permutation representation associated to the action of S_4 on the set of edges of the cube.

3. Let V be the standard representation of S_5.

(a) Find the expression of $V \otimes V$ as a direct sum of irreducible representations of S_5.

(b) Which terms in your answer to part (a) lie in $\mathrm{Sym}^2(V)$, and which ones lie in $\wedge^2 V$?

4. Let $p \geq 3$ be a prime, and consider the dihedral group $D_p = \langle r, s \,|\, r^p = 1,\ s^2 = 1,\ srs^{-1} = r^{-1} \rangle$ (where r is the rotation by $2\pi/p$ and s is any reflection).

(a) Given χ the character of *any* 2-dimensional representation of D_p, what do the defining relations of D_p tell you about $\chi(r)$ and $\chi(s)$?

(b) Determine the character table of D_p.

5. Determine the character tables of the two non-abelian groups of order 8, i.e. the dihedral group D_4, and the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (where $i^2 = j^2 = k^2 = ijk = -1$). How do they compare?

(Hint: start with the 1-dimensional representations).

6. Let S be a finite set on which a finite group G acts, let V be the corresponding permutation representation of G (recall V has a basis $\{e_s\}_{s \in S}$, and $g \in G$ acts by $g \cdot e_s = e_{g \cdot s}$).

(a) Show that the multiplicity in V of the trivial representation U (i.e. the number of copies of U appearing in the decomposition of V) is equal to the number of orbits of the G-action on S. In particular, if the action of G on S is transitive, then we can write $V = U \oplus V'$, where V' does not contain U as a subrepresentation.

(Note: this can be done either by calculating $H(\chi_U, \chi_V)$ and using Burnside's lemma, or directly by finding the invariant subspace V^G).

(b) Suppose that G acts transitively on S, and $|S| \geq 2$. We say that the action is *doubly transitive* if for every $s_1, s_2, s_1', s_2' \in S$ with $s_1 \neq s_2$ and $s_1' \neq s_2'$, there exists $g \in G$ such that $s_1' = g \cdot s_1$ and

$s_2' = g \cdot s_2$. First check that this is equivalent to the statement that the action of G on $S \times S$ has exactly two orbits; then show that the representation V' considered in part (a) is irreducible if and only if the action of G on S is doubly transitive.

Hint: first show that the permutation representation for the action of G on $S \times S$ is $V \otimes V$, and show that $H(\chi_U, \chi_{V\otimes V}) = H(\chi_V, \chi_V)$. (What property of χ_V does this use?)

7. Fulton-Harris exercise 2.21: prove that the orthogonality of the rows of the character table implies an orthogonality property for the columns (assuming the fact that there are as many rows as columns). Specifically, show that:

(i) For $g \in G$,

$$\sum_{\chi} |\chi(g)|^2 = \frac{|G|}{c(g)},$$

where the sum is over all irreducible characters, and $c(g)$ is the number of elements in the conjugacy class of g. (Note that for $g = e$ this reduces to $|G| = \sum \dim(V_i)^2$).

(ii) If $g, h \in G$ are not conjugate, then $\sum_{\chi} \overline{\chi}(g)\, \chi(h) = 0$.

8. Fulton-Harris exercise 2.34: let V and W be irreducible representations of G, and $L_0 : V \to W$ any linear map. Define $L : V \to W$ by

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot v).$$

Show that $L = 0$ if V and W are not isomorphic, and that if $V = W$ then L is multiplication by $\mathrm{tr}(L_0)/\dim(V)$.

9*. Fulton-Harris exercise 2.37: show that if V is a faithful representation of G, i.e. $\rho : G \to GL(V)$ is injective, then any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.

Hint: denoting by U the trivial representation of G and by V_i any irreducible representation, what is the largest term in $\dim \mathrm{Hom}_G(V_i, (V \oplus U)^{\otimes n})$ as $n \to \infty$? (Why can't one just consider $V^{\otimes n}$?)

10. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?