In the previous lecture, you found that the work done by gravity is always equal to:

\[ W_{by\, F_{grav}} = -mg\Delta h \]

That expression on the right-hand side should look familiar: it looks like the change in the potential energy due to gravity! In other words, we seem to have:

\[ W_{by\, F_{grav}} = -\Delta U_{grav} \]

where \( \Delta U \) is equal to \( U_{final} - U_{initial} \). In this lecture, we will see that this relationship is quite general. Whenever there is a potential energy for some interaction, there will also be a corresponding force associated with that interaction. So far we’ve seen only two kinds of potential energy:

Gravitational PE: \( U_{grav} = mgh \) associated with gravitational force

Elastic (spring) PE: \( U_{elastic} = \frac{1}{2}kx^2 \) associated with spring force

These kinds of forces are known as conservative forces, because they arise from the potential energy of an interaction. All other forces (normal, tension, friction, drag) are nonconservative forces: there is no potential energy \( U \) associated with these forces. This distinction between conservative and nonconservative forces will be extremely important, so you should memorize this (short!) list of conservative forces and assume that all other forces are nonconservative.\(^1\)

For any conservative force, it turns out that the work done by that force is:

\[ W_{by\, F} = -\Delta U \]

Indeed, the relationship between potential energy and conservative forces is even deeper: given the potential energy for any conservative force, we can derive the force law for that force. As we will see in lecture, the vector components of the force are given by:

\[ F_x = -\frac{\partial U}{\partial x} \quad \quad F_y = -\frac{\partial U}{\partial y} \]

---

\(^1\) Interestingly, all the fundamental forces (electromagnetism, gravity, strong force, and weak force) are conservative, but most of the macroscopic forces that we experience (normal, tension, friction, drag) are nonconservative.
Physical Sciences 2: Lecture 4c

These funny curly “d”’s represent partial derivatives. If you haven’t seen them before (and most of you probably haven’t), just know that \( \frac{\partial U}{\partial x} \) means “take the derivative of \( U \) with respect to \( x \), assuming that all other variables are constant.” What do we mean by that? Consider a simple example. If \( U = xy \), then the usual derivative would be calculated by using the product rule:

Traditional (“total”) derivative: \( \frac{dU}{dx} = y + x \frac{dy}{dx} \)

while the partial derivative would treat \( y \) as a constant, so the second term vanishes:

Partial derivative: \( \frac{\partial U}{\partial x} = y \)

Although partial derivatives are not usually introduced until the 3rd semester of calculus or later, they are much easier to compute than traditional (“total”) derivatives.

For instance, if we choose conventional coordinates, the gravitational potential energy will be \( U_{grav} = mg \) (because the height is just equal to \( y \)). Then the components of the gravitational force will be:

\[
F_x = -\frac{\partial U}{\partial x} = 0 \quad F_y = -\frac{\partial U}{\partial y} = -mg
\]

which is exactly what we would expect.

If there’s more than one kind of potential energy, then the negative total change in potential \( U \) equals the work done by all the associated conservative forces: \( -\Delta U_{total} = W_{by \ all \ conservative \ F} \). For the two kinds of potential energy we’ve seen so far, we can write this as:

\[
-\Delta U_{grav} - \Delta U_{elastic} = W_{by \ F_{grav}} + W_{by \ F_{spring}}
\]

Using these relationships between work and potential energy for conservative forces, we’ll be able to derive an important relationship between work and mechanical energy:

**Total work done by all nonconservative forces = Change in mechanical energy**, or:

\[
W_{by \ all \ nonconservative \ F} = \Delta E_{mech} = \Delta K + \Delta U
\]

Furthermore, if a system is isolated so the total energy is conserved, we will find that:

\[
W_{by \ all \ nonconservative \ F} = -\Delta E_{internal}
\]

This equation really helps us to understand the relationship between work and energy. It says that any nonconservative force that does positive work will require a decrease in the internal energy (of something!). For instance, if you lift a box, you are doing positive work on the box.
Physical Sciences 2: Lecture 4c

The force of you on the box is a nonconservative force, so the total *internal energy of something must decrease*—in this case, it’s *your* internal energy (ATP!). If a car accelerates, the static friction force does positive work, so some kind of internal energy must decrease—in this case, it’s the energy stored in the gasoline. Conversely, if there’s a nonconservative force doing *negative* work, then the internal energy (of something!) must increase: the most common example would be friction or drag causing an increase in the thermal energy of a system. This equation doesn’t tell you where (or how) the internal energy will change, but it says that some kind of internal energy *must* change if the system is isolated and there’s a nonconservative force doing some non-zero work. This equation also helps us understand how a roller coaster rolling on a frictionless track can have its mechanical energy conserved. The normal force of the track acts on the car, so the car is not isolated. But this normal force always acts *perpendicular* to the displacement of the car so it does no work—and hence doesn’t change the energy of the car.

Many physical systems can be called “machines”: a machine converts one form of energy into another, often by doing some kind of work. (You are a machine.) Most machines are not limited in the total amount of work they can do, but are instead limited by their *power*: the *rate* at which they can convert energy. Your muscles have a limited rate of converting ATP into mechanical work; a car has a limited rate of converting chemical energy into mechanical work. The rate of energy conversion $\Delta E/\Delta t$, or the *rate of work* $dW/dt$ will tell you the power involved in that process. The SI unit for power is the watt (1 watt = 1 joule per second).

Finally, we will see some *potential energy diagrams* that show $U$ for a system as a function of position. A classic example is the “roller-coaster” diagram shown at right. In these diagrams, the force is given by $-\partial U/\partial x$. At point (b) the derivative is negative, so there is a force pointing in the *positive* x-direction: the car will experience a force to the right (which should be obvious). Points (a), (c), and (d) are points of equilibrium where the force is zero. But at (a) and (d) the equilibrium is unstable: if you nudge the car a little bit, it will roll away from equilibrium. At (c) however the car is at stable equilibrium: a tiny nudge will give rise to a restoring force that will push the car back to equilibrium. These points of stable equilibrium will turn out to be quite important and universal, because any system near a point of stable equilibrium will oscillate around that equilibrium point—just like a mass oscillates back and forth on a spring.
Physical Sciences 2: Lecture 4c

- Learning objectives: After this lecture, you will be able to...

1. Calculate the work done by gravity on an object during some process.

2. Distinguish between conservative and non-conservative forces.

3. Understand the relationship between the potential energy of a conservative force and the (vector) components of that force.

4. Understand the relationship between the work done by a conservative force and the change in potential energy associated with that force.

5. Understand the relationship between the change in mechanical energy for any system and the work done by all non-conservative forces on that system.

6. Use any of the relationships derived in Module 4 between work, force, kinetic energy, and potential energy to calculate the work done by some force on an object, or to calculate other physical quantities (force, distance, speed, etc.).

7. Calculate the power required to convert energy from one form to another, or the power required to do a certain amount of work.
Work and Potential Energy

• Highlights from the pre-reading:

In the previous lecture, you showed that the work done by gravity on an object is always given by: \( W_{\text{by } \vec{F}_{\text{grav}}} = -mg\Delta h \). But the gravitational potential energy is \( U = mgh \). So:

\[
W_{\text{by } \vec{F}_{\text{grav}}} = -\Delta U_{\text{grav}}
\]

In general, for a potential energy \( U \), the work done by its force \( \vec{F} \) is: \( W_{\text{by } \vec{F}} = -\Delta U \).

Given a potential energy \( U \) for some interaction, the (vector) components of the force associated with that interaction are equal to (minus) the partial derivatives of \( U \):

\[
F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y}
\]

Using these relationships, along with the others developed earlier, you can calculate the work done by a force, and the magnitudes of various forces, speeds, and displacements.

It is important to distinguish between work and power: the rate of doing work. For most machines (including you!), the key limitation is power, not the total amount of work.

The relationship between force and potential energy can be seen in a potential energy diagram. With these diagrams, you can locate points of stable equilibrium.

• Learning objectives: After this lecture, you will be able to...

1. Calculate the work done by gravity on an object during some process.

2. Distinguish between conservative and non-conservative forces.

3. Understand the relationship between the potential energy of a conservative force and the (vector) components of that force.

4. Understand the relationship between the work done by a conservative force and the change in potential energy associated with that force.

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7. Calculate the power required to convert energy from one form to another, or the power required to do a certain amount of work.
\[ \Delta K = \int \vec{F}_{\text{net}} \cdot d\vec{r} = W_{\text{net}} = W_{F_1} + W_{F_2} + \ldots \]

**Work done by gravity (pre-video)**

- Recall the basic definition of work done by a constant force:
  \[ W_{\text{by } F} = \vec{F} \cdot \Delta \vec{r} \]

1. Write the dot product in terms of components; recall the components of \( \Delta r \) are \((\Delta x, \Delta y)\).
   \[
   \vec{F} = (F_x, F_y), \quad \Delta \vec{r} = (\Delta x, \Delta y)
   \]
   \[ \vec{F} \cdot \Delta \vec{r} = F_x \Delta x + F_y \Delta y \]

2. Choose conventional \((x, y)\) axes and find an expression for the work done by gravity on an object of mass \( m \). The answer could depend on \( m, x, y, \) and/or \( g \):
   \[ W_{\text{by } F_{\text{grav}}} = 0 + (-mg)\Delta y = -mg \Delta y \]

3. What is the relationship between the **work done by gravity** and the **change in the gravitational potential energy**, \( \Delta U_{\text{grav}} \)? (Hint: what is \( U_{\text{grav}} \)?)
   \[ \Delta U_{\text{g}} = U_f - U_i = mg(h_f - h_i) = mg \Delta h = mg \Delta y \]
   \[ W_g = -\Delta U_{\text{g}} \]

4. Show that the same relationship found in part (3) will hold even if you choose non-conventional coordinates, such as in the diagram shown at right.
   \[ \Delta U_{\text{g}} = mg \Delta h = mg(h_f - h_i) = mg(-L \cos \theta) \]
   \[ W_g = \vec{F}_g \cdot \Delta \vec{r} = mg(c \cos \theta, -s \sin \theta) \cdot (L, 0) = mg \cos \theta L = -\Delta U_{\text{g}} \]
   \[ \vec{F}_g = mg (\cos \theta, -\sin \theta) \]
Am I getting it?

- A box is at rest on a ramp. You push the box up the ramp (with friction), and stop at the top of the ramp. You then turn around and push the box back down to where it started. The box ends at rest.

Consider the work done on the box.

For the first half of this process (pushing the box up the ramp):

1. The work done by the normal force (of the ramp on the box) is:
   - a) negative
   - b) zero
   - c) positive

2. The work done by gravity is:
   - a) negative
   - b) zero
   - c) positive

3. The work done by kinetic friction is:
   - a) negative
   - b) zero
   - c) positive

4. The work done by you is:
   - a) negative
   - b) zero
   - c) positive

For the entire process (pushing the box up and then back down again):

5. The work done by the normal force (of the ramp on the box) is:
   - a) negative
   - b) zero
   - c) positive

6. The work done by gravity is:
   - a) negative
   - b) zero
   - c) positive

7. The work done by kinetic friction is:
   - a) negative
   - b) zero
   - c) positive

8. The work done by you is:
   - a) negative
   - b) zero
   - c) positive

\[ W_g = -\Delta U_g = -mg \Delta h \text{ with } \Delta h = 0 \]

\[ \vec{F} \cdot \Delta \vec{r} = \text{down} - \vec{F} \cdot (-\Delta \vec{r}) = 2\vec{F} \cdot \Delta \vec{r} \]
Activity 1: Potential energy, Work, and Force

- We found that: $W_{by F_{grav}} = -\Delta U_{grav}$. It turns out that for any conservative force...
- The work done by that force is equal to minus the change in potential energy.
- For every "kind" of potential energy, there is an associated force:

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y}$$

- Besides gravity, we have seen one other conservative force: the force of a spring. If a spring stretches along the $x$-axis, and the equilibrium position is $x_{eq}$, the potential energy of the spring is related to the spring constant $k$ as:

1. Given the spring potential energy below, find an expression for the $x$-component of the force exerted by a spring.

$$U_{elastic} = \frac{1}{2} k (x - x_{eq})^2$$

$$F_x = -\frac{\partial U_{elastic}}{\partial x} = -\frac{\partial}{\partial x} \left[ \frac{1}{2} k (x - x_{eq})^2 \right]$$

$$= -\frac{1}{2} k \frac{\partial}{\partial x} [(x - x_{eq})^2] \quad \text{Hooke's Law}$$

$$\Rightarrow F_{spring} = -\frac{1}{2} k \frac{2}{2} (x - x_{eq}) = -k (x - x_{eq})$$

2. On the axes below, sketch the potential energy of a spring, and the $x$-component of the spring force. (For simplicity, take the equilibrium position to be $x_{eq} = 0$.)

3. What do you notice about the point on the potential energy graph where the force is zero?

Slope is zero for $U$ when $F_x = 0$

- Bonus! An object (mass $m$) hangs vertically from a spring. Set $y = 0$ when the spring is not stretched or compressed. Write an expression for the total potential energy of the system (including spring and gravity forces), and find where the PE is at a minimum.

$$U = U_{spring} \quad U = \frac{1}{2} ky^2 + mgy$$

P.E. at min where $\frac{\partial U}{\partial y} = 0$

$$\Rightarrow ky + mg = 0 \quad y = -\frac{mg}{k}$$
Conservative and Non-conservative forces (pre-video)

1. We must distinguish **conservative forces**, which are related to some kind of potential energy, and **nonconservative forces**, which don’t have any associated potential energy.

List all the kinds of **conservative forces** we have encountered so far in this course:
(Hint: there are only two)

\[ \vec{F}_g \iff U_g \quad \vec{F}_{\text{spring}} \iff U_{\text{spring}} \]

List all the kinds of **nonconservative forces** we have encountered:

Friction, pull, tension, normal, drag

- On the previous page, you found: \( W_{\text{grav}} \) F = \(-\Delta U_{\text{grav}}\). It turns out that this is true in general for **conservative forces**:
  - For every “kind” of potential energy, there is an associated **conservative force**.
  - The work done by that force is equal to minus the change in potential energy.

We won’t prove this in general, but we have shown it for gravity.

2. If an object moves a distance \( D \), and then comes back to where it started, show that the total work done by a **conservative force** must be zero.

\[ W_g = -\Delta U = (U_f - U_i) = U_f - U_i = U(x_f) - U(x_i) = 0 \]

3. If an object moves a distance \( D \), and then comes back to where it started, show that the total work done by a **nonconservative force can be nonzero**. (Find an example for each kind of nonconservative force you listed above.)

\[ \Delta E_{\text{mech}} = \Delta K + \Delta U = W_{c} + W_{\text{NC}} - W_{c} = W_{\text{NC}} \]

\[ \Delta K = W_{\text{net}} = \sum F \cdot \Delta x = W_{c} + W_{\text{NC}} \]

\[ \Delta U = -W_{c} \]

\[ \Delta E_{\text{mech}} = W_{\text{NC}} \]
Activity 2: Non-Conservative Forces

1. Given the following scenario, list all forces on the skier. Which forces are non-conservative?

A m=50-kg skier is at the top of a slope with a height of h=20 m and an angle of 20° from the horizontal. Starting from rest, she skis straight down the hill. At the base of the hill, she is traveling at a speed of v=15.5 m/s. The coefficient of kinetic friction between her skis and the hill is \( \mu_k = 0.03 \); she is affected by both kinetic friction and air drag.

![Diagram of forces: \( F_g \), \( F_N \), \( F_{k_f} \), \( F_{\text{drag}} \), \( F_r \)]

- From the previous page (pre-video) we know:

  Total work done by all nonconservative forces = Change in mechanical energy, or:

  \[
  W_{by\ all\ nonconservative} = \Delta E_{\text{mech}} = \Delta K + \Delta U
  \]

2. Use the statement above, to solve for the work done by drag on the skier in terms of \( m \), \( g \), \( h \), \( v \), and the work done by kinetic friction \( W_{Kf} \) (you don't need to work out the numbers).

3. \[
W_{by\ all\ nonconservative} = \text{Sum of all Work from non conservative forces}
\]

4. \[
W_{Kf} + W_{\text{drag}} = \Delta K + \Delta U
\]

4. \[
W_{\text{drag}} = \frac{1}{2}mv^2 - mgh - W_{Kf}
\]

4. \[
W_{Kf} = \frac{F_{Kf}}{2} \Delta \theta
\]

4. \[
= -\mu_k mg \cos \theta \cdot \frac{h}{\sin \theta}
\]

4. \[
= -800 J
\]

Bonus! Carry out all the calculations from question (3) above.
Activity 3: Kinesin, ATP, work, and power

1. **If a system is isolated**, show: The work done by all **nonconservative** forces \( = -\Delta E_{\text{internal}} \)

\[ \Delta E_{\text{tot}} = 0 \]

\[ \Rightarrow \Delta K + \Delta U + \Delta E_{\text{int}} = 0 \Rightarrow W_{NC} + \Delta E_{\text{int}} = 0 \]

\[ W_{NC} = -\Delta E_{\text{int}} \]

2. The relationship from question (1) above is very useful in thinking about work and energy. What can you say about the change in internal energy for the following cases:

- A car slows down due to air drag.

\[ W_{NC} = W_{\text{drag}} \quad \Delta E_{\text{int}} \]

- A car accelerates due to the static friction force between the car and the road.

\[ W_{NC} = W_{\text{st}} \quad \Delta E_{\text{int}} \]

- You stand up from your chair.

\[ W_{NC} = W_N \quad \Delta E_{\text{int}} \quad \text{ATP} \]

3. Kinesin exerts a force of 7 pN (piconewton = \( 10^{-12} \) N) as it pulls a vesicle over a distance of 8 nm. Is kinesin doing positive or negative work? What change in internal energy is required? (Note: Hydrolysis of one molecule of ATP releases \( 6 \times 10^{-20} \) J.)

\[ W_{\text{pull}} = F_{\text{pull}} \cdot \Delta \vec{r} \]

\[ F_{\text{pull}} = 7 \times 10^{-12} N \]

\[ 8 \text{ nm} \Rightarrow \rightarrow 8 \times 10^{-9} m = 5.6 \times 10^{-20} J \]

4. The **power** required for a given conversion of energy is equal to the amount of energy converted divided by the time required: \( P = \Delta E / \Delta t \). Compared with walking slowly up the stairs, if I **run** up the stairs, what is the difference in:

- the amount of work required \( W_{NC} = \Delta E_{\text{mech}} = mgh \) *Same!*

- the power required \( P = \frac{\Delta E}{\Delta t} \) *More power running*

*Bonus!* The instantaneous power of a given force is the “rate of work,” \( dW/dt \). Show that the instantaneous power of a force \( \vec{F} \) on an object with velocity \( \vec{v} \) is: \( P = \vec{F} \cdot \vec{v} \).