Plan for today

1. More on Wannier functions and quantum spin Hall bands

2. Time reversal invariant topological insulators
   - 2d theory via edge states
   - 3d theory - general & powerful discussion

QSH

\[ C_+ = C \]
\[ C_- = -C \]

\[ \Rightarrow \text{no net Chern} \]

\[ \Rightarrow \text{we can construct Wannier functions} \]
\[ \text{but they will not carry definite spin } \frac{1}{2}. \]

Suppose we use these \( \frac{1}{2} \) non-symmetric
Wannier basis to build a tight binding model:

\[ H = -\sum_{\beta=\alpha} E_{\beta} \mathbf{c}_{\alpha}^\dagger \mathbf{c}_{\beta} + h.c \]

We know that \( S^z_{\text{tot}} \) commutes with \( H \):

\[ \{ S^z_{\text{tot}}, H \} = 0. \]

But it cannot be that

\[ S^z_{\text{tot}} = \sum_R S^z_R \text{ with } \]

\[ \{ S^z_R, S^z_{R'} \} = 0 \]

If this were true, then each \( S^z_R \) will transform with definite \( S^z \) under \( U(\text{spin}(1)) \) which we know is impossible.
Though $S^z_{tot}$ is conserved, it must be expressed in terms of operators that involve electrons at different sites.

Note that in $k$-space

$$S^z_{tot} = \sum_k c^\dagger_k \sigma^z_k c_k$$

is a sum of fermion bilinears in $k$-space.

In real space, it will also be a sum of fermion bilinears.

In general with $S^z_{tot} = \sum_{RR', \alpha \beta} O^{\alpha \beta}_{RR'} c^\dagger_{RR'} c_{RR'}$, $O^{\alpha \beta}_{RR'}$ is necessarily non-zero for some $R' \neq R$. 
This form of $S_{\text{tot}} \Rightarrow$ under a $U(\text{spin}(1))$ rotation by angle $\theta$, in real space $c^\dagger_{\mathbf{k} \beta}$ mix with $c^\dagger_{\mathbf{k}' \beta}$ for some $\mathbf{k}' \neq \mathbf{k}$.

**Edge states of quantum spin Hall effect** Consider an edge at $x = 0$.

Counter propagating edge modes $\uparrow$ spin $\uparrow \downarrow$ spin form one-way edge modes moving in opposite directions.

In formal terms, the edge Hamiltonian.
$H_{\text{edge}} = \int dy \; \psi_e^+ (-i \partial_{by}) \psi_e + \psi_e^+ (i \partial_{by}) \psi_e$

Define $\psi_e = \left[ \begin{array}{c} \psi_{e+} \\ \psi_{e-} \end{array} \right]$ to write

$H_{\text{edge}} = \int dy \; \psi_e^+ (-i \sigma^z \partial_{by}) \psi_e$

$\sigma^z = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]$ — standard massless Dirac fermions in 1D.

Edge dispersion

Though this is a non-chiral edge, it
is protected from becoming gapped by a symmetry.

The global symmetry of the system is $U(1) \times U(1)$. Assuming $U(\text{charge})$ is preserved, the mass terms for the Dirac fermion take the form

$$\int d\gamma \, m_x \psi^\dagger \sigma^2 \psi + m_y \psi^\dagger \sigma^y \psi$$

But either of these breaks $U(\text{spin})$.

$\Rightarrow$ so long as we have $U(1) \times U(1)$

global symmetry, there is no mass term that can gap out the edge $\Rightarrow$ gaplessness of the edge.
is symmetry protected. QSH state: example of "symm. protected topological insulator".

Time reversal protected 2d top. insulator

Start with the QSH state & explore breaking even $V_{\text{spin}}(1)$ with a generic spin-orbit coupling - but $T$-reversal is preserved.

In real space $\mathcal{S} \equiv \mathbb{C}_p \rightarrow \mathbb{C}_z$

$\mathbb{C}_z \rightarrow -\mathbb{C}_z$

so that $\mathcal{S}^2 = -1$ when acting on a single electron.

Now as $S^z$ is not conserved we
cannot talk about a DSH effect any more; but consider the edge theory.

I act on $\psi_e$ then

$I: \psi_e \to \psi_e$

$\psi_e \to -\psi_e^\dagger$

It is easy to check that $\psi_e^\dagger \sigma_e^x \psi_e$ and $\psi_e^\dagger \sigma_e^y \psi_e$ are both $I$-odd

$\Rightarrow$ both possible mass terms are disallowed by $I$-reversal.

Gepleness of edge is protected by $U(1)$ and $I$-reversal.

This edge structure defines a
distinct 2d band insulator
- "the topological insulator" in 2d

Time reversal protected 3d topological
insulators

Consider an insulator in 3d whose
excitations consist only of electrons
and their composites (i.e. there is
no "exotic" thing like fractional
charge).

We know that excitations have
charge \( q = ne \) with \( n \in \mathbb{Z} \),
2 that \( n \) even corresponds to
bosons
2 n odd to fermions.
Consider a "gedanken" exp't where, as a probe, we place a magnetic monopole $M$ (of basic strength $2\pi/e$) inside the medium [Note this is not an excitation but an external probe].

Time reversal & monopoles: Electric fields are even & mag. fields are odd under $T$.

$\Rightarrow$ electric charge $q_e$ is even, mag. $q_m$ is odd.

Suppose that monopole $M$ nucleates some electric charge $q_e$. 
It's time reversed partner TM must also have the same charge \( q_e \).

Imagine creating & bringing together \( M \) & TM to get rid of all the mag. charge but with total electric charge is \( 2q_e \).

But once the mag. charge is gone, the result must be an excitation.
of the insulator that we are probing. All excitations have electric charge $ne$, $n \in \mathbb{Z}$

$\Rightarrow \quad 2q_e = ne$

$\Rightarrow \quad q_e = 0, e, 2e, 3e, \ldots$

or $q_e = \frac{e}{2}, \frac{3e}{2}, \frac{5e}{2}, \ldots$

Thus there are 2 distinct solutions

$q/e = 0 \quad \text{mod} \ 2$

or $q/e = \frac{1}{2} \quad \text{mod} \ 2$

Thus $J$-reversed invariant insulators with no exotic excitations come in 2 distinct varieties corresponding
to probe e-charge on monopole
(either 0 or $\frac{1}{2}$ mod 2).

In an ordinary insulator, probe
monopole has e-charge $= 0$ mod 2.

We see that it is possible to have
a distinct 3d $\mathbb{Z}$-reversal protected
top. insulator where probe monopole
has charge $q e / e = \frac{1}{2}$ mod 2.

Note: If $\mathbb{Z}$ is broken, $M$ & $\mathbb{M}$
do not need to have same e-charge
& we cannot conclude anything
on the charge of $M$; thus it
is $J$ that protects the
Quantization of e-charge of $\mathcal{M}$

② Monopole is not an excitation, hence could have fractional e-charge.

③ Quantization of monopole e-charge guarantees stability to perturbations (e.g., interactions, disorder, ...)

What kind of insulator supports such $q_e = e/2$ charged monopoles?

To understand, consider in greater detail the bound state of $\mathcal{M}$ and $\bar{\mathcal{M}}$, where both have charge $e/2$. 
The classical angular momentum is

\[ \vec{L} = (e_1 g_2 - e_2 g_1) \hat{\vec{R}} \]

\[ = \left( \left( \frac{e}{2} + \frac{\hbar}{2e} \right) - \left( \frac{e}{2} \right) \right) \hat{\vec{R}} \]

\[ = \frac{\hbar}{2} \hat{\vec{R}} \]

(Where \( \vec{R} \) = separation between the 2 particles).

This suggests that the bound state of \( M = TM \) (when both have e-charge \( \frac{e}{2} \)) has a \( \frac{1}{2} \)-integer spin.

More precise in the HW: the bound state has "orbital angular momentum" of \( \frac{1}{2} \).

Quantum mechanically, the Berry
phase seen when \( M \) moves in a loop is

\[
\left( \frac{1}{2} \times \frac{1}{2\pi} \right) (\text{solid angle}) = \frac{1}{2} (\text{solid angle})
\]

\[\text{TM}\]

[Note: For e-change \( \frac{1}{2} \) moving around a strength \( \frac{1}{2} \) monopole, the Berry phase is \( \frac{1}{2} \) (solid angle); for a e-change \( \frac{1}{2} \), the Berry phase is \( \frac{1}{4} \) (solid angle) but here the monopole also moves around the e-change \( \frac{1}{2} \) of TM which gives another \( \frac{1}{4} \) (solid angle)]

This Berry phase is same as a unit e-change moving around
a strength $2\pi$-monopole $\Rightarrow$ ground state has internal ang. mom. 1/2.

Further under $J$, $M \leftrightarrow TM$

$\Rightarrow$ relative rotor $\vec{R} \rightarrow -\vec{R}$

In the ground state (projecting to the spin-$1/2$ doublet),

$P \bar{P} \vec{P} \rightarrow \vec{S} =$ spin-$1/2$ operator

$J : \vec{S} \rightarrow -\vec{S}$

$\Rightarrow$ bound state is a Kramer's doublet under $J$. (See HW2).