We finally apply the Borsuk-Ulam Theorem to graph theory and proper colorings.

**Definition.**

The Kneser Graph $KG_{n,k}$ for $n \geq 2$, $k \geq 1$ is given by
- vertices are subsets of $\{1,2,\ldots,n\}$ of size $k$.
- given two vertices $S$ and $T$ (as sets), there is an edge $ST$ iff $S$ and $T$ are disjoint.

**Examples**

$n=4$

- $k=1$
  - \begin{align*}
  &\{1\} \rightarrow \{2\} \rightarrow \{3\} \rightarrow \{4\} \rightarrow \{1\} \\
  &\{2\} \rightarrow \{3\} \rightarrow \{4\} \rightarrow \{1\} \rightarrow \{2\} \\
  &\{3\} \rightarrow \{4\} \rightarrow \{1\} \rightarrow \{2\} \rightarrow \{3\} \\
  &\{4\} \rightarrow \{1\} \rightarrow \{2\} \rightarrow \{3\} \rightarrow \{4\}
  \end{align*}

$n=5$

- $k=2$
  - \begin{align*}
  &\{1,2\} \rightarrow \{3,4\} \rightarrow \{5\} \rightarrow \{1,2\} \\
  &\{3,4\} \rightarrow \{5\} \rightarrow \{1,2\} \rightarrow \{3,4\} \\
  &\{5\} \rightarrow \{1,2\} \rightarrow \{3,4\} \rightarrow \{5\} \rightarrow \{1,2\} \\
  &\{1,2\} \rightarrow \{3,4\} \rightarrow \{5\} \rightarrow \{1,2\} \rightarrow \{3,4\}
  \end{align*}

- $k=3$
  - \begin{align*}
  &\{1,2,3\} \rightarrow \{4,5\} \rightarrow \{1,2,3\} \\
  &\{4,5\} \rightarrow \{1,2,3\} \rightarrow \{4,5\} \rightarrow \{1,2,3\} \\
  &\{1,2,3\} \rightarrow \{4,5\} \rightarrow \{1,2,3\} \rightarrow \{4,5\} \rightarrow \{1,2,3\}
  \end{align*}

- $k=4$

- $n=4$
  - $\chi(KG_{4,1}) = 4$
  - $\chi(KG_{4,2}) = 2$
  - $\chi(KG_{4,4}) = 1$ if $k \geq \frac{n}{2}$

- $n=5$
  - $\chi(KG_{5,1}) = 3$

**Theorem (Lovász, 1978; Conjectured by Kneser in 1955 as an "exercise")**

The chromatic number of the Kneser Graph $KG_{n,k}$ is $n - 2k + 2$. 

**References:**

We first prove, algorithmically, that $n-2k+2$ suffice.

Then, to prove that $n-2k+2$ color are necessary, we will use the Borsuk-Ulam theorem.

Proof of upper bound

We give a procedure to color $KG_{n,k}$ with $n-2k+2$ colors:

- Color all the sets that include 1 with the first color.
- Color all the remaining sets that include $n-2k+1$ with the $n-2k+1$-st color.
- Color all the remaining vertices with the $n-2k+2$-nd color.

We need to show that this coloring is proper.

- For the first $n-2k+1$ colors, all the vertices have an element of their corresponding sets in common, so they can't share an edge.

- For the remaining vertices, they are subsets of $\{n-2k+2, \ldots, n\}$ of size $k$. This is equivalent to subsets of $\{1, \ldots, 2k-1\}$ of size $k$. However, we have seen that $\chi(KG_{n,k}) = 1$ if $k' \geq n' = 2k-1$. Since this is the case here, the same color can be used for all subsets of size $k$ of $\{n-2k+2, \ldots, n\}$.

**Lemma** (Equivalent to Borsuk-Ulam Theorem)

If $S^n$ is covered by $n+1$ subsets $X_1, \ldots, X_{n+1}$ such that each of them is either open or closed, then at least one of them contains a pair of antipodal points.
We are now left with proving that \( n - 2k + 2 \) are necessary.

Proof of lower bound (Greene, 1982)

We proceed by contradiction, assuming that one can color \( K_{n,k} \) with \( d := n - 2k + 1 \) colors. Then, there exists a proper coloring

\[ c : \binom{n}{k} \rightarrow \{1, \ldots, d\}. \]

Let \( X \) be a set of \( n \) points on \( S^d \) in general position, i.e., such that no \( d + 1 \) points lie on the same equator of \( S^d \).

These \( n \) points correspond to the elements of \( \{1, 2, \ldots, n\} \) used to define the vertices, so that a vertex corresponds to a set of \( k \) points of \( S^d \).

Construct \( d \) open sets \( U_1, \ldots, U_d \) as follows:

for a point \( x \in S^d \), consider all the points of \( X \) in the same hemisphere as \( x \) (in the closest half-sphere from \( x \)). For each subset \( V \) of \( k \) points in the same hemisphere, \( x \in U_{c(V)} \). Note that the sets need not to be disjoint.

Construct the closed set \( F_d \) = \( S^d \setminus \{U_1, \ldots, U_d\} \). \( U_1, \ldots, U_d \) cover \( S^d \) using only open and closed sets, so by the Borsuk-Ulam theorem, one of them contains antipodal points.

Call this set either \( U_i \) (i.e., \( i = 1, \ldots, d \)) or \( F_d \).

If \( U_i \) contains antipodal points, \( x \) and \(-x\); then, each hemisphere contains \( k \) points of \( x \) corresponding to vertices \( v \) and \( v' \), both colored with color \( i \).

Also, \( v \) and \( v' \) correspond to two disjoint sets of \( k \)
vertices, so they must be adjacent in $K_{G, n, k}$.

However, $c(v) = c(v')$, which means that the coloring is not proper.

So the set containing antipodal points must be $F_{d+1}$.

If $x \in F_{d+1}$, then the open hemisphere around $x$ does not contain $k$ elements of $X$. The same is true for $-x$.

Therefore, the equator (for the poles of $S^d$ $x$ and $-x$) contains at least $n - 2(k-1) = n - 2k + 2 = d + 1$. This contradicts the fact that $X$ is in general position.

Hence, it is not possible to color $K_{G, n, k}$ with $d = n - 2k + 1$ colors.

\[\square\]

**Remark**

There also exists a purely combinatorial proof of the Lovász

Theorem, using Tucker's Lemma. (see for example [Lon13, §2.1])

**References:**

[Ma03, §3.3]

[Lon13, §2.1]