Math 108 - Geometric Combinatorics

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The Kneser Conjecture

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We finally apply the Borsuk-Ulam Theorem to graph theory and proper colorings

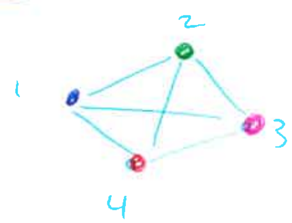
Definition.

The Kneser Graph $KG_{n,k}$ for $n \geq 2, k \geq 1$ is given by

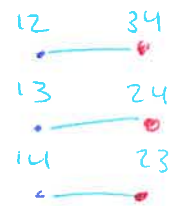
- vertices are subsets of $\{1,2,\cdots,n\}$ of size k.
- given two vertices S and T (as sets), there is an edge ST iff S and T are disjoints.

Examples

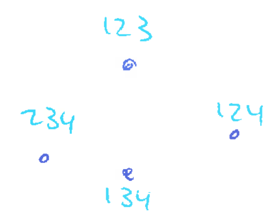
$n=4$
$k=1$



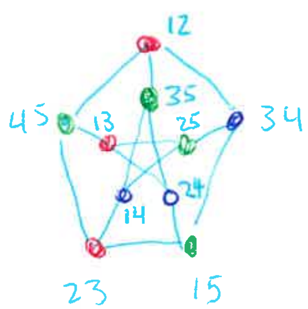
$n=4$
$k=2$



$n=4$
$k=3$



$n=5$
$k=2$



$\chi(KG_{4,1}) = 4$

$\chi(KG_{4,2}) = 2$

$\chi(KG_{n,k}) = 1$ if $k > \frac{n}{2}$

$\chi(KG_{5,2}) = 3$

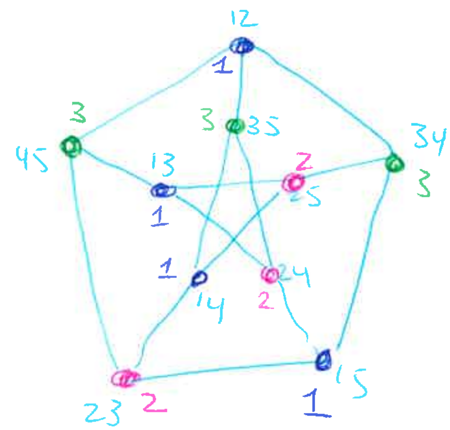
Theorem (Lovasz, 1978; Conjectured by Kneser in 1955 as an "exercise")

The chromatic number of the Kneser Graph $KG_{n,k}$ is $n-2k+2$.

We first prove, algorithmically, that $n-2k+2$ suffice. Then, to prove that $n-2k+2$ color are necessary, we will use the Borsuk-Ulam theorem.

Proof of upper bound

We give a procedure to color $KG_{n,k}$ with $n-2k+2$ colors:



- Color all the sets that include 1 with the first color

  ...

- Color all the remaining sets that include $n-2k+1$ with the $n-2k+1$-st color
- Color all the remaining vertices with the $n-2k+2$-nd color.

We need to show that this coloring is proper.

- For the first $n-2k+1$ colors, all the vertices have an element of their corresponding sets in common, so they can't share an edge.
- For the remaining vertices, they are subsets of $\{n-2k+2, \dots, n\}$ of size k. This is equivalent to subsets of $\{1, \dots, 2k-1\}$ of size k. However, we have seen that $\chi(KG_{n',k'}) = 1$ if $k' > n' = 2k-1$. Since this is the case here, the same color can be used for all subsets of size k of $\{n-2k+2, \dots, n\}$. $\blacksquare$

Lemma (Equivalent to Borsuk-Ulam theorem)

If S^n is covered by $n+1$ subsets $X_1, \dots, X_{n+1}$ such that each of them is either open or closed, then at least one of them contains a pair of antipodal points.

We are now left with proving that $n-2k+2$ are necessary.

Proof of lower band (Greene, 2002)

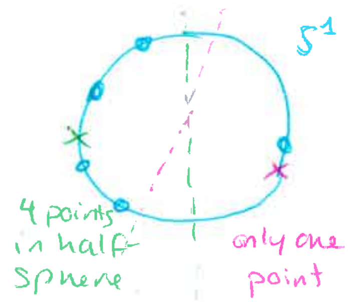
We proceed by contradiction, assuming that one can color $KG_{n,k}$ with $d := n-2k+1$ colors. Then, there exists a proper coloring

$$c : \binom{n}{k} \longrightarrow \{1, \dots, d\}.$$

Let X be a set of n points on S^d in general position, i.e. such that no $d+1$ points lie on the same equator of S^d.

These n points correspond to the elements of $\{1, 2, \dots, n\}$ used to define the vertices, so that a vertex correspond to a set of k points of S^d.

Construct d open sets $\mathcal{U}_1, \dots, \mathcal{U}_d$ as follows:

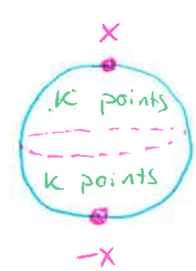


for a point $x \in S^d$, consider all the points of X in the same open hemisphere as x (in the closest half-sphere from x). For each subset v of k points in the same hemisphere, $x \in \mathcal{U}_{c(v)}$. Note that the sets need not to be disjoint.

Construct the closed set $\mathcal{F}_{d+1} = S^d \setminus \{\mathcal{U}_1, \dots, \mathcal{U}_d\}$.

$\mathcal{U}_1, \dots, \mathcal{U}_d, \mathcal{F}_{d+1}$ cover S^d using only open and closed sets, so by the Borsuk-Ulam theorem, one of them contains antipodal points:

Call this set either $\mathcal{U}_i$ ($i \in 1, \dots, d$) or $\mathcal{F}_{d+1}$.



If $\mathcal{U}_i$ contains antipodal points, x and $-x$: then, each hemisphere contains k points of X corresponding to vertices v and v', both colored with color i.

Also, v and v' correspond to two disjoint sets of k

vertices, so they must be adjacent in $KG_{n,k}$.

However, $c(v) = i = c(v')$, which means that the coloring is not proper.

So the set containing antipodal points must be $\mathcal{F}_{d+1}$.

If $x \in \mathcal{F}_{d+1}$, then the open hemisphere around x does not contain k elements of X. The same is true for $-x$.

Therefore, the equator (for the poles of S^d x and $-x$) contains at least $n - 2(k-1) = n-2k+2 = d+1$. This contradicts the fact that X is in general position.

Hence, it is not possible to color $KG_{n,k}$ with $d = n-2k+1$ colors. $\blacksquare$

Remark

There also exists a purely combinatorial proof of the Lovasz Theorem, using Tucker's Lemma. (see for example [Lon13, §2.1])

References:

[Mat03, §3.3]

[Lon13, §2.1]