In this lecture, we saw the following two examples of functions, $f : [n] \to [2n]$ such that

$$f(x) = 2x$$

and $g : [n] \to \{\text{even}, \text{odd}\}$ where

$$g(x) = \begin{cases} \text{even} & \text{if } x \text{ is even} \\ \text{odd} & \text{if } x \text{ is odd.} \end{cases}$$

Recall that $[n] = \{1, 2, \ldots, n\}$.

We can also interpret f and g as subsets of the Cartesian product, $[n] \times [2n]$ and $[n] \times \{\text{even}, \text{odd}\}$ respectively. In particular, we can identify f with the set F defined as

$$F = \{(a, b) \mid a \in [n], b \in [2n], 2a = b\}$$

and g with the set G defined as

$$G = \{(a, b) \mid a \in [n], b \in \{\text{even}, \text{odd}\}, a \text{is } b\}.$$

Sometimes thinking about functions in this way will make it easier to prove properties of these functions.

We can also ask if f and g are injective, or surjective. A relation from a set A to a set B is injective if every element $b \in B$ appears in at most one pair in the relation, and is surjective if every element $b \in B$ appears in at least one pair. To show that a relation is surjective, we can consider the set $f^{-1}(b) = \{a \mid f(a) = b\}$, and prove that this set has size at least 1 for every $b \in B$. To show that a relation is injective, we can show that if $f(a) = f(b)$, then $a = b$. To show that a relation is injective, we can also show that the set $f^{-1}(b) = \{a \mid f(a) = b\}$ has size at most 1 for every $b \in B$, but this is sometimes harder to do.

The following theorems answer the question as to which of f and g are injective or surjective.

Theorem 1. *The function $f : [n] \to [2n]$ such that $f(x) = 2x$ is injective but not surjective.*

Proof. If $f(a) = f(b)$, then $2a = 2b$, and thus $a = b$. Therefore, f is injective.

On the other hand, f is not surjective. To prove this we just need a counterexample, that is, an element $b \in B$ such that the set $f^{-1}(b)$ is empty. One example is 1, and in particular, the set $\{a \mid f(a) = 1\}$ is empty. □

Theorem 2. *Let n be an integer at least 3. The function $g : [n] \to \{\textit{even}, \textit{odd}\}$ where*

$$g(x) = \begin{cases} \textit{even} & \textit{if } x \textit{ is even} \\ \textit{odd} & \textit{if } x \textit{ is odd.} \end{cases}$$

surjective but not injective.

Proof. There are only two elements in the range, even and odd, so to prove that the function f is surjective, we can just consider them one by one. The set $g^{-1}(\text{even}) = \{a \mid g(a) = \text{even}\}$ contains the element 2 because 2 is even, and $g^{-1}(\text{odd}) = \{a \mid g(a) = \text{odd}\}$ contains the element 1 because 1 is odd. Thus, g is surjective.

On the other hand, g is not injective. To prove this, we can give a counterexample to the statement that if $g(a) = g(b)$, then $a = b$. In particular, $g(1) = g(3)$, as both 1 and 3 are odd, but it is not true that $1 = 3$.

Another way to see that g is not injective is to note that $g^{-1}(\text{odd}) = \{a \mid g(a) = \text{odd}\}$ has size at least 2. □