1. Let \( R \subset \mathbb{C}[x] \) be the subring of polynomials \( P \) such that the coefficient of \( x \) in \( P \) is zero.

   (a) (1 point) Give an embedding of \( \text{Spec } R \) into \( \mathbb{A}^2 \), and show that the image has a cusp.

   **Solution:** Note that \( R \) is generated as a \( \mathbb{C} \)-algebra by the polynomials \( x^2 \) and \( x^3 \), which satisfy \((x^3)^2 = (x^2)^3\). There is thus a surjection
   \[
   f : \mathbb{C}[u, v]/(u^3 - v^2) \to R
   \]
   sending \( u \) to \( x^2 \) and \( v \) to \( x^3 \). We claim that this map is an isomorphism. Indeed, every element of the left hand side can be uniquely represented as a polynomial in \( u \) and \( v \) without any terms of degree at least 2 in \( v \). Therefore, as a vector space, \( \mathbb{C}[u, v]/(u^3 - v^2) \) has a basis given by the \( u^i \) and the \( u^i v \). As \( f(u^i) = x^{2i} \) and \( f(u^i v) = x^{2i+3} \), this basis is sent by \( f \) to the basis of \( R \) consisting of all powers of \( x \) except for \( x \) itself, which shows that \( f \) is an isomorphism.

   Now the map
   \[
   \mathbb{C}[u, v] \to \mathbb{C}[u, v]/(u^3 - v^2) \cong R
   \]
   shows that \( \text{Spec } R \) is isomorphic to the plane curve \( u^3 = v^2 \), which has a cusp at the origin.

   (b) (1 point) Find a smooth curve \( \text{Spec } S \) with a map \( \text{Spec } S \to \text{Spec } R \) which is an isomorphism on topological spaces. Observe that this means that the composition \( \text{Spec } S \to \text{Spec } R \to \mathbb{A}^2 \) is a closed embedding of topological spaces but not a closed embedding of algebraic varieties.

   **Solution:** Note that we have a map \( R \to \mathbb{C}[x] \), which gives a map of varieties \( \mathbb{A}^1 \to \text{Spec } R \). To see that this map is an isomorphism on points, we look at the composition
   \[
   g : \mathbb{A}^1 \to \text{Spec } R \to \mathbb{A}^2
   \]
defined by
\[ x \mapsto (x^2, x^3). \]
We know that \( \text{Spec } R \) embeds into \( \mathbb{A}^2 \) as the vanishing locus of \( u^3 - v^2 \), so it suffices to show that every \( (u,v) \) with \( u^3 - v^2 = 0 \) can be uniquely expressed as \( g(x) \). If \( v = 0 \), then \( u = 0 \), and \( g \) sends only 0 to \((0,0)\). On the other hand, if \( v \neq 0 \), then \( x = \frac{u}{v} \) is the unique point sent to \((u,v)\), as desired.

As \( \mathbb{A}^1 \) and \( \text{Spec } R \) both have the cofinite topology (as they are both curves), it follows that the map \( \mathbb{A}^1 \to \text{Spec } R \) is also an isomorphism of topological spaces. Thus the map \( \mathbb{A}^1 \to \mathbb{A}^2 \) is a closed embedding of topological spaces, but is not a closed embedding of varieties because the corresponding map of algebras \( \mathbb{C}[u,v] \to R \to \mathbb{C}[x] \) is not surjective, as it factors through \( R \).

2. (2 points) Let \( R \) be a finite type \( \mathbb{C} \)-algebra that is integral (i.e., has no zero-divisors.) Let \( S \) be a multiplicative system in \( R \). Show that the localization \( R_S \) is a finite type \( \mathbb{C} \)-algebra if and only if it is isomorphic to the localization \( R_f \) at a single nonzero element \( f \). (Recall that \( R_f \) is the localization of \( R \) at the multiplicative system \( \{1, f, f^2, \cdots\} \).)

**Solution:** First we show that \( R_f \) is finitely generated. Let \( R \) be generated as an algebra by elements \( f_1, \cdots, f_n \). Then \( R_f \) will be generated by \( f_1, \cdots, f_n, \frac{1}{f} \), so is also finite type.

On the other hand, assume \( R_S \) is a finite type algebra. Then it is generated by elements \( \frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n} \), with \( b_i \in S \). Let \( f \) be the product of the \( b_i \), which will still be an element of \( S \). Each \( \frac{a_i}{b_i} \) can be written as a fraction with denominator \( f \), so all polynomials in those elements can be written as fractions with denominators powers of \( f \). Thus, every element of \( R_S \) lies inside \( R_f \), as desired.

3. Our definition of \( \text{Spec } R \) as a topological space still makes sense for rings \( R \) which are not finite type \( \mathbb{C} \)-algebras. We will not worry too much about such algebras in this class, but let us briefly discuss the case of \( \mathbb{R} \)-algebras.

(a) (1 point) Classify the maximal ideals of \( \mathbb{R}[x] \), and describe the map
\[ \text{Spec}(\mathbb{C}[x]) \to \text{Spec}(\mathbb{R}[x]). \]

**Solution:** As \( \mathbb{R}[x] \) is a principal ideal domain, the maximal ideals of \( \mathbb{R}[x] \) will be those generated by one irreducible polynomial. Thus, we get one maximal ideal \( (x - a) \) for every real number \( a \) and one maximal ideal \( (x^2 + ax + b) \) for every quadratic polynomial with no real roots (equivalently, for every pair of conjugate non-real complex numbers.)

The map \( f : \mathbb{C}[x] \to \mathbb{R}[x] \) sends an ideal \( I \) to its intersection with \( \mathbb{R}[x] \). It is clear that if \( r \) is real, \( f \) sends \((x - r)\) to \((x - r)\). On the other hand, if \( r \) is
non-real, then any polynomial with real coefficients and root \( r \) must also have \( \bar{r} \) as a root and hence be a multiple of \( (x - r)(x - \bar{r}) \). Thus, \( f \) sends \( (x - r) \) for non-real \( r \) to \( (x - r)(x - \bar{r}) \).

(b) (1 point) Classify the maximal ideals of \( \mathbb{R}[x, y]/(x^2 + y^2 + 1) \), and describe the map

\[
\text{Spec}(\mathbb{C}[x, y]/(x^2 + y^2 + 1)) \to \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1)).
\]

Note that the vanishing locus of \( x^2 + y^2 + 1 = 0 \) in \( \mathbb{R}^2 \) is empty, and yet we can still study the algebraic geometry of this ring.

**Solution:** Let \( g \) denote the map

\[
\text{Spec}(\mathbb{C}[x, y]/(x^2 + y^2 + 1)) \to \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1)).
\]

We start by claiming that \( g \) is surjective. Indeed, let \( m \) be a maximal ideal of \( \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1)) \). Then \( m + \mathbb{R}i m \) is a non-unit ideal in \( \text{Spec}(\mathbb{C}[x, y]/(x^2 + y^2 + 1)) \), and is thus contained in some maximal ideal \( m' \). As \( m' \) contains (the image of) \( m \), we see that \( g(m') \) must be a maximal ideal containing \( m \), hence equaling \( m \).

We know that the maximal ideals of \( \text{Spec}(\mathbb{C}[x, y]/(x^2 + y^2 + 1)) \) are of the form \( ((a - x), (y - b)) \) for \( a, b \) complex numbers with \( a^2 + b^2 + 1 = 0 \). Equivalently, we can describe this ideal as containing exactly the polynomials that vanish at \( (a, b) \). As a polynomial with real coefficients vanishes at \( (a, b) \) if and only if it vanishes at \( (\bar{a}, \bar{b}) \), we see that \( g((x - a), (y - b)) = g((x - \bar{a}), (y - \bar{b})) \).

Conversely, we will show that if \( g((x - a), (y - b)) = g((x - c), (y - d)) \), then either \( (c, d) = (a, b) \) or \( (c, d) = (\bar{a}, \bar{b}) \). This will imply that maximal ideals of \( \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1)) \) are classified by pairs \( (a, b) \) with \( a^2 + b^2 + 1 = 0 \), modulo conjugation. Assume that \( (c, d) \) is neither \( (a, b) \) nor \( (\bar{a}, \bar{b}) \). Then there are real numbers \( r \) and \( s \) such that \( rc + sd \) is equal to neither \( ra + sb \) nor \( r\bar{a} + sb \). Then the polynomial \( (rx + sy - ra - sb)(rx + sy - r\bar{a} - s\bar{b}) \) has real coefficients and vanishes at \( (a, b) \) but not at \( (c, d) \). This gives an element of \( g((x - a), (y - b)) \) that is not in \( g((x - c), (y - d)) \).

4. Let \( S \) be a subset of \( \mathbb{Z}^n \) containing 0 and closed under addition (in other words, a sub-semigroup of \( \mathbb{Z}^n \)). We can define a ring \( \mathbb{C}[S] \) whose elements are formal linear combinations \( \sum a_i t^{s_i} \) with the \( s_i \in S \), with multiplication determined by the rule \( t^{s_i} \cdot t^{s_j} = t^{s_i + s_j} \). An affine toric variety is the spectrum of a ring \( \mathbb{C}[S] \). Toric varieties give a large family of easy examples of varieties.

(a) (1 point) Show that every inclusion \( S \subseteq S' \) gives a map of toric varieties \( \text{Spec} \mathbb{C}[S'] \to \text{Spec} \mathbb{C}[S] \).
Solution: There is a map of algebras \( \mathbb{C}[S] \to \mathbb{C}[S'] \) sending \( t \) to \( t' \). Taking \( \text{Spec} \) gives us the desired map.

(b) (1 point) Show that any toric variety has an open subset which is isomorphic to a torus (i.e., the spectrum of an algebra \( \mathbb{C}[x_i, x_i^{-1}] \)). This is why these varieties are called toric.

Solution: Let \( S' \) be the group generated by \( S \). Then as \( S \) is a subgroup of \( \mathbb{Z}^n \), it must be isomorphic to \( \mathbb{Z}^m \) for some \( m \). It follows that \( \mathbb{C}[S'] \) is isomorphic to an algebra \( \mathbb{C}[x_1, \ldots, x_m, x_i^{-1}, \ldots, x_m^{-1}] \), and hence has spectrum a torus. It remains to show that the map \( \text{Spec} \mathbb{C}[S'] \to \text{Spec} \mathbb{C}[S] \) is an open embedding, or equivalently that the map of algebras \( \mathbb{C}[S] \to \mathbb{C}[S'] \) is a localization. (Technically one needs that it is a localization by one element, but this follows assuming \( S \) is finitely generated (a necessary assumption for the problem) by Problem 2.)

Note that the set of elements of the form \( t \) is a multiplicative system in \( \mathbb{C}[S] \). Inverting these elements gives \( \mathbb{C}[S'] \), as desired.

5. (2 points) Recall in class that we mentioned that \( X = \mathbb{A}^2 - \{(0,0)\} \) is not an affine variety. More precisely, we claim that there is no affine variety \( Y \) with a map \( \pi : Y \to \mathbb{A}^2 \) and two open subvarieties \( U \) and \( V \) satisfying the following properties:

- \( Y \) is the union of \( U \) and \( V \)
- \( \pi \) induces an isomorphism of varieties between \( U \) (respectively, \( V \)) and the complement of the \( x \)-axis (respectively, the \( y \)-axis) in \( \mathbb{A}^2 \)
- \( \pi \) induces an isomorphism of varieties between the intersection \( U \cap V \) and the locus where \( xy \) does not vanish in \( \mathbb{A}^2 \).

Prove this. (Hint: One way of doing this is to think about maps from such a variety \( Y \) to \( \mathbb{A}^1 \).)

Solution: Maps from a variety \( Y \) to \( \mathbb{A}^1 \) are in bijection with elements in \( \mathcal{O}(Y) \), so we will work in the language of regular functions. By our assumptions, \( \mathcal{O}(U) \cong \mathbb{C}[x, y, y^{-1}] \) and \( \mathcal{O}(V) \cong \mathbb{C}[x, y, x^{-1}] \). We also have \( \mathcal{O}(U \cap V) \cong \mathbb{C}[x, y, x^{-1}, y^{-1}] \).

Each regular function on \( Y \) corresponds to a pair of regular functions, one from each of \( \mathcal{O}(U) \) and \( \mathcal{O}(V) \), whose restrictions to \( \mathcal{O}(U \cap V) \) agree. From our computations above, we see that this implies that \( \mathcal{O}(Y) \cong \mathbb{C}[x, y] \). As \( Y \) is affine, this implies that \( Y \cong \mathbb{A}^2 \). But this is a contradiction, as then the origin would be in the image of \( \pi \).
6. (1 point) Look up the definition of a sheaf. Use google to find as many motivations as you can for why you would define such an object. Elaborate on the one you find most convincing.

**Solution:** Many possible answers.