Last time: began a digression on categories. (then we'll return to (bi)linear algebra)

Def: A *category* is a collection of *objects* + for each pair of objects, a collection of *morphisms* $\mathrm{Mor}(A,B)$, and an operation called *composition* of morphisms,
$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \to \mathrm{Mor}(A,C) \quad \text{st.}$$
$$f \,,\, g \longmapsto g \circ f$$
1) every object A has an *identity* morphism $id_A \in \mathrm{Mor}(A,A)$ st. $\forall f \in \mathrm{Mor}(A,B)$, $f \circ id_A = id_B \circ f = f$.

2) composition is *associative*: $(f \circ g) \circ h = f \circ (g \circ h)$.

★ **Products and sums in categories:**

- Given objects A, B in a category $\mathcal{C}$, a *product* $A \times B$ is an object Z of $\mathcal{C}$ and a pair of maps $\pi_1 : Z \to A$, $\pi_2 : Z \to B$ st. $\forall T \in \mathrm{ob}\,\mathcal{C}$, $\forall f_1 \in \mathrm{Mor}(T,A)$, $f_2 \in \mathrm{Mor}(T,B)$, $\exists!$ (unique) $\varphi \in \mathrm{Mor}(T,Z)$ st. $\pi_1 \circ \varphi = f_1$ and $\pi_2 \circ \varphi = f_2$.

$$\begin{tikzcd} & T \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists!\varphi"] \arrow[dr, bend left, "f_2"] & \\ A & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Ex: in Sets, $Z = A \times B$ usual Cartesian product, π_1, π_2 projection maps; given $f_1 : T \to A$, $f_2 : T \to B$, def. $\varphi : \begin{array}{l} T \to A \times B \\ t \mapsto (f_1(t), f_2(t)) \end{array}$

Same in Groups, Vect_k

- A *sum* of objects A and B is an object Z of $\mathcal{C}$ + maps $i_1 : A \to Z$, $i_2 : B \to Z$ st. $\forall T \in \mathrm{ob}\,\mathcal{C}$, $\forall f_1 \in \mathrm{Mor}(A,T)$, $\forall f_2 \in \mathrm{Mor}(B,T)$, $\exists!\, \varphi \in \mathrm{Mor}(Z,T)$ st. $\varphi \circ i_1 = f_1$ & $\varphi \circ i_2 = f_2$.

$$\begin{tikzcd} & T & \\ A \arrow[ur, bend left, "f_1"] \arrow[r, "i_1"'] & Z \arrow[u, dashed, "\varphi"] & B \arrow[ul, bend right, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

Ex: in Sets, this is $Z = A \sqcup B$ disjoint union; define $\varphi : Z \to T$, $x \mapsto \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B. \end{cases}$

in Vect_k, it's $Z = A \oplus B$ (so... sum = product!) with i_1, i_2 = inclusion of $\begin{array}{l} A \text{ as } A \oplus 0 \subset Z \\ B \quad\;\; 0 \oplus B \subset Z \end{array}$ define $\varphi : Z \to T$, $(a,b) \mapsto f_1(a) + f_2(b)$.

★ **Functors:**

Def: C, D categories. A (covariant) *functor* $F : C \to D$ is an assignment
- to each object X in C, an object $F(X)$ in D.
- to each morphism $f \in \mathrm{Mor}_C(X,Y)$, a morphism $F(f) \in \mathrm{Mor}_D(F(X), F(Y))$

st. 1) $F(id_X) = id_{F(X)}$ 2) $F(g \circ f) = F(g) \circ F(f)$.

Ex: 1) forgetful functor taking a group, a top. space, ... to the underlying set.

2) on vector spaces, given a vect. space V, $F: W \longmapsto \mathrm{Hom}(V,W)$

if $f: W \to W'$ is linear, then induced map $\mathrm{Hom}(V,W) \xrightarrow{F(f)} \mathrm{Hom}(V,W')$

This gives a functor $\mathrm{Vect}_k \to \mathrm{Vect}_k$ (denoted $\mathrm{Hom}(V,\cdot)$) $\quad a \longmapsto f\circ a$.

3) Complexification, $\mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{C}}$: on objects, $V \mapsto V_{\mathbb{C}}$, on morphisms $\varphi \mapsto \varphi_{\mathbb{C}}$ seen last time

* A contravariant functor = same except direction of morphisms is reversed:

$f \in \mathrm{Mor}_{\mathcal{C}}(X,Y) \mapsto F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(Y), F(X))$; $F(g\circ f) = F(f)\circ F(g)$.

Ex: on Vect_k, $V \mapsto V^*$ dual (see HW5).

* There's one more layer to this, if you love category theory: given 2 functors $F, G: \mathcal{C} \to \mathcal{D}$, a natural transformation t from F to G is the data, $\forall X \in \mathrm{ob}\,\mathcal{C}$, of a morphism $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$, s.t. $\forall X, Y \in \mathrm{ob}\,\mathcal{C}$, $\forall f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$,

$$\begin{array}{ccc} F(X) & \xrightarrow{t_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{t_Y} & G(Y) \end{array}$$

commutes in $\mathcal{D}$.

Ex: on Vect_k, $V \mapsto V^{**}$ double dual is a (covariant) functor. We've said there is a "natural" map $ev_V: V \to V^{**}$, $v \mapsto (\ell \mapsto \ell(v))$ (isom. if $\dim < \infty$)

The precise meaning is: ev_V is part of a natural transformation of functors $\mathrm{Vect}_k \to \mathrm{Vect}_k$, from the identity functor to the double dual functor.

Bilinear forms:

Def: A bilinear form on a vector space V over field k is a map $b: V\times V \to k$ that is linear in each variable: $\forall u, v, w \in V$, $\forall \lambda \in k$,
$$\begin{cases} b(\lambda v, w) = b(v, \lambda w) = \lambda b(v,w) \\ b(u+v, w) = b(u,w) + b(v,w) \\ b(u, v+w) = b(u,v) + b(u,w). \end{cases}$$

This is not a linear map $V \times V \to k$ $\quad (b(\lambda(v,w)) = b(\lambda v, \lambda w) = \lambda^2 b(v,w) \neq \lambda b(v,w))$.

Def: We say b is symmetric if $b(v,w) = b(w,v)$ $\forall v, w \in V$

skew-symmetric if $b(v,w) = -b(w,v)$

Ex: • the usual dot product on k^n, $(v,w) \mapsto \sum_{i=1}^n v_i w_i$ is symmetric.

• $b: k^2 \times k^2 \to k$, $b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1$ $\left(= \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right)$ is skew symmetric

★ Given a bilinear map $b: V \times V \to k$, we get a linear map $\varphi_b: V \to V^*$ by defining $\varphi_b(v) = b(v,\cdot) \in V^*$ (maps $w \in V$ to $b(v,w) \in k$).

Conversely, $\varphi: V \to V^*$ determines $b(v,w) = (\varphi(v))(w)$ bilinear form.

This defines a bijection $B(V) \xrightarrow{\sim} \mathrm{Hom}(V,V^*)$.

Def: The rank of $b: V \times V \to k$ is the rank of $\varphi_b: V \to V^*$ ($= \dim \mathrm{Im}\, \varphi_b$).
If φ_b is an isomorphism, say b is nondegenerate.

★ For a given vector space V, $B(V) = \{\text{bilinear forms } V \times V \to k\}$ is a vector space over k. What is its dimension?

If we choose a basis $\{e_1 \dots e_n\}$ for V, it is enough to specify $b(e_i,e_j)$ $\forall i,j$ in order to determine b: by bilinearity, $b(\sum_i x_i e_i, \sum_j y_j e_j) = \sum_{i,j} x_i y_j b(e_i,e_j)$.

The values of $b(e_i,e_j)$ can be chosen freely - eg. a basis of $B(V)$ is given by $(b_{kl})_{\substack{1 \le k \le n \\ 1 \le l \le n}}$ $\quad b_{kl}(e_i,e_j) = \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise}. \end{cases}$

So: $\dim B(V) = (\dim V)^2$ (consistent with $B(V) \simeq \mathrm{Hom}(V,V^*)$!)
The bijection $b \mapsto \varphi_b$ is an isom. of vector spaces !)

★ Given a basis $\{e_1 .. e_n\}$ of V, $b: V \times V \to k$ is represented by an $n \times n$ matrix $a_{ij} = b(e_i,e_j)$

$$b\Big(\sum_i x_i e_i, \sum_j y_j e_j\Big) = \sum_{i,j} x_i y_j b(e_i,e_j) = (x_1 \dots x_n)\, A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

matrix of b; $a_{ij} = b(e_i,e_j)$

so: in terms of column vectors, $\underline{\underline{b(X,Y) = X^T A Y}}$.

★ Remark: The isomorphism $\begin{matrix} B(V) & \xrightarrow{\sim} & \mathrm{Hom}(V,V^*) \\ b & \longmapsto & \varphi_b \end{matrix}$ is natural, in the sense that

SKIP THIS REMARK IF YOUR HEAD HURTS

we have contravariant functors $V \mapsto B(V)$ and $V \mapsto \mathrm{Hom}(V,V^*)$,
(on morphisms, $f: V \to W \rightsquigarrow \begin{matrix} B(f): B(W) & \longrightarrow & B(V) \\ b(\cdot,\cdot) & \longmapsto & b(f(\cdot),f(\cdot)) \end{matrix}$ and $\begin{matrix} \mathrm{Hom}(W,W^*) \to \mathrm{Hom}(V,V^*) \\ \varphi \mapsto f^t \circ \varphi \circ f \end{matrix}$)

and the isom's $B(V) \xrightarrow{\sim} \mathrm{Hom}(V,V^*)$ define a natural transformation between them.

* **Def:** If $S \subset V$ is a subspace of a vector space equipped with a bilinear form $b: V \times V \to k$, we define its <u>orthogonal complement</u>
$S^\perp = \{w \in V / b(v,w) = 0 \ \forall v \in S\}$. This is a vector subspace.

(Equivalently: $S^\perp = \mathrm{Ann}(\varphi_b(S))$: $\varphi_b(S) = \{b(v,\cdot),\ v \in S\} \subset V^*$
$\mathrm{Ann}(\varphi_b(S)) \subset V$ vectors on which all these linear forms vanish.)

⚠ This is most useful if b is symmetric or skew. Otherwise we have to worry about left-orthogonal vs. right-orthogonal.

* If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$ (else: $\dim V - \dim \varphi_b(S)$).

Ex: • $V = \mathbb{R}^n$ with the standard dot product $b(v,w) = \sum_{i=1}^n v_i w_i$: then
$V = S \oplus S^\perp$ the "usual" orthogonal complement
because: $S \cap S^\perp = \{0\}$ (if $v \in S \cap S^\perp$ then $b(v,v) = 0 \Rightarrow v = 0$).
and $\dim S + \dim S^\perp = \dim V$.
true in $\mathbb{R}^n$, not necess other fields k^n!!

• <u>but</u> for $b: k^2 \times k^2 \to k$
$$b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1$$ (skewsymmetric, nondegenerate)
$S \subset k^2$ 1-dim! subspace spanned by any nonzero vector $v \Rightarrow \underline{S^\perp = S}$!!
(because $b(v,w) = 0 \Leftrightarrow \det(v,w) = 0 \Leftrightarrow w \in k \cdot v = S$).

<u>Inner product spaces:</u>

Def: V vector space over $\mathbb{R}$. We say a bilinear form $b: V \times V \to \mathbb{R}$ is an <u>inner product</u> if (1) b is symmetric, and
(2) $\forall v \in V$, $b(v,v) \geq 0$, and $b(v,v) = 0$ iff $v = 0$.
Say b is <u>positive definite</u>.

This definition only makes sense over an <u>ordered</u> field so "$b(v,v) \geq 0$" makes sense. In practice this means $\mathbb{R}$. We can't define an inner product over $\mathbb{C}$, because $b(iv,iv) = i^2 b(v,v) = -b(v,v)$ $\Rightarrow$ cannot hope for positivity of a bilinear form.

To fix this, there's a trick: observe $|\lambda|^2 \geq 0 \ \forall \lambda \in \mathbb{C}$!

Def: V vector space over $\mathbb{C}$, a <u>Hermitian form</u> is a map $h: V \times V \to \mathbb{C}$ which is linear in the second variable, and conjugate linear (or "complex antilinear") in the first variable: $h(\lambda v, w) = \bar\lambda h(v,w) \ \forall \lambda \in \mathbb{C}$ vs. $h(v,\lambda w) = \lambda h(v,w)$.
$h(v_1+v_2, w) = h(v_1,w) + h(v_2,w)$ $\quad$ $h(v, w_1+w_2) = h(v,w_1) + h(v,w_2)$
+ <u>conjugate symmetric</u>: $h(v,w) = \overline{h(w,v)}$.

(same convention as Artin, opposite of Axler's).

Will then study $\mathbb{C}$-vector spaces with Hermitian inner product = positive-definite Hermitian form