Physical Sciences 2

Physical Sciences 2: Homework #2: Forces, Impulse, and Kinematics
Solution Key

After completing this homework, you should…
- Be able to use both definitions of impulse interchangeably
- Be able to calculate impulse from a graph of force vs. time
- Know Newton’s second and third laws
- Use dimensional analysis to identify proper units for an unknown quantity
- Use dimensional analysis to derive forms of equations
- Know the kinematic equations and when/how to use them
- Be able to interpret graphs of position vs. time, velocity vs. time, and accelerations vs. time
- Understand the conditions on acceleration of an object during free fall and during projectile motion
Here are summaries of this lecture’s important concepts to help you complete this homework:

**Impulse**

- When two objects collide, they experience a force over a period of time that changes the momentum.
- Impulse $J$ is the change in momentum:

\[ J = \Delta p = \int_{t_1}^{t_2} F_{\text{net}} \, dt = \langle F_{\text{net}} \rangle \Delta t \]

- On a force vs. time curve, impulse is the area under the curve.

**Kinematics**

- Interpreting Graphs
  - Position vs. time
    \[ \text{slope of } s \text{ vs. } t \text{ gives } v \]

- Velocity vs. time
  \[ \text{slope of } v \text{ vs. } t \text{ gives } a \]
  \[ \text{area under curve gives the change in position } \Delta s \]

- Acceleration vs. time
  \[ \text{area under gives the change in velocity } \Delta v \]
Kinematics

- in the special case of constant acceleration, the following equations relate an object's position, velocity, and acceleration at any time:

\[
\begin{align*}
\vec{V} &= \vec{V}_0 + \vec{a} \cdot t \\
\vec{r} &= \vec{r}_0 + \vec{v}_0 \cdot t + \frac{1}{2} \vec{a} \cdot t^2 \\
\vec{V}^2 &= \vec{V}_0^2 + 2\vec{a} \cdot (\vec{r} - \vec{r}_0)
\end{align*}
\]

- in the case of free fall, the only acceleration is due to gravity, so

\[
\begin{align*}
a_x &= 0 \\
a_y &= -g = -9.81 \text{ m/s}^2
\end{align*}
\]

- this means that an object's velocity in the x-direction will not change

\[V_x = \text{constant}\]

- typical questions:
  - for an object fired with initial velocity \(\vec{V}_0\) at an angle \(\theta\) to the ground,
    - how high does the object rise?
    - how much time passes before the object reaches the highest point of its trajectory?
    - how much time is the object in the air?
    - what is the total horizontal distance traveled by the object?

- all of these questions can be answered by using some combination of the kinematic equations
1. Projectile Motion Tutorial (1 pt)

A projectile is fired at an angle \( \theta \) from the ground at an initial velocity of \( v_0 \).

a) What is the velocity of the projectile in the \( x \)-direction? Express your answer in terms of \( \theta \) and \( v_0 \) as well as any fundamental constants.

To get the \( x \)-component of the velocity, we simply multiply \( v_0 \) by \( \cos \theta \), to find

\[
v_{0,x} = v_0 \cos \theta
\]

b) What is the velocity of the projectile in the \( y \)-direction? Express your answer in terms of \( \theta \) and \( v_0 \) as well as any fundamental constants.

To get the \( y \)-components of the velocity, we simply multiply \( v_0 \) by \( \sin \theta \), to find

\[
v_{0,y} = v_0 \sin \theta
\]

c) How long does it take until the projectile has reached the highest point in its trajectory? Express your answer in terms of \( \theta \) and \( v_0 \) as well as any fundamental constants.

At the highest point of its trajectory, the \( y \)-component of its velocity is equal to 0 because the object is no longer moving upwards. As the particle is moving through its trajectory, the only acceleration acting on it is in the \( y \)-direction and is due to gravity. With the initial \( y \)-velocity from Part b) and these facts, we can use a kinematic equation to solve for the amount of time necessary for the particle to reach the highest point:

\[
v_y = v_{0,y} + at
\]

\[
0 = v_0 \sin \theta - gt
\]

\[
t = \frac{v_0 \sin \theta}{g}
\]
d) How long does it take until the projectile reaches the ground? Express your answer in terms of $\theta$ and $v_0$ as well as any fundamental constants.

The trajectory of the projectile is symmetric. This means it will take the same amount of time for the projectile to go from the ground to the top of its trajectory as for it to go from the top of its trajectory to back to the ground. The reason for this is because gravity is the only acceleration acting on the object. With this information, we can write the total amount of time the projectile is in the air as

$$t_{total} = 2 \frac{v_0 \sin \theta}{g}$$

e) How far does the projectile travel? Express your answer in terms of $\theta$ and $v_0$ as well as any fundamental constants.

The distance the projectile travels is dependent on its $x$-velocity and how long it is in the air. Because the only acceleration is due to gravity and is in the $y$-direction, the $x$-velocity does not change. To get the distance traveled by the projectile, we simply multiply the projectile’s $x$-velocity by the amount of time the object is in the air, and simplify using the trigonometric identity $2\sin\theta\cos\theta = \sin2\theta$.

$$x = x_0 + v_{0,x}t$$

$$x = 0 + (v_0 \cos \theta) \left(2 \frac{v_0 \sin \theta}{g} \right)$$

$$x = \frac{2v_0^2 \sin \theta \cos \theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$
2. Putting Everything Together (Exam-Type Question): Adam and Betty (1 pt)

Adam and Betty are studying physics together:

Adam: “I think Newton’s 3rd law is wrong.”
Betty: “….ummm….why?”
Adam: “Well, think about someone walking. I know there’s an equal and opposite force between the ground and me. That makes the net force zero, so I shouldn’t go anywhere, but I clearly am able to walk places.”
Betty: “Ahhh, actually Newton’s still right because….”

What flaw in Adam’s reasoning did Betty find? (Maximum 2-3 sentences.)

Adam has forgotten that the forces related by Newton’s 3rd law act on different objects. The net force on the Earth + Adam system is zero, but the net force acting on Adam is nonzero.

3. Putting Everything Together: It’s a drag (1 pts)

There are two different “laws” for the drag force on an object moving through a fluid such as air or water. Using dimensional arguments, we can deduce the form of these two laws, but not which law is valid in any given situation.

In both laws, the magnitude of the drag force can depend on the size of the object (ℓ, a length) and on the speed (v, the magnitude of the velocity) of the object.

a) One of the expressions for drag force depends on the viscosity of the fluid (denoted by η, the Greek letter eta). Viscosity has SI units of kg·m⁻¹·s⁻¹. Determine the expression for the magnitude of the drag force that depends on the viscosity of the fluid. Your answer should be in the form of a proportionality, i.e. \( F_{\text{drag}} \propto \ell^a v^b \eta^c \) where the exponents a, b, and c are chosen so that the dimensions work out correctly.

First, let’s write down the dimensions of the relevant quantities:

\[
\begin{align*}
[F] &= [ML/T^2] \\
[\ell] &= [L] \\
v &= [L/T] \\
[\eta] &= [M/LT]\quad\text{(from the units given)}
\end{align*}
\]

Then we can write down the dimensions of the equation \( F_{\text{drag}} \propto \ell^a v^b \eta^c \):

\[
\begin{align*}
\left[\frac{ML}{T^2}\right] &= [L]^a \left[\frac{L}{T}\right]^b \left[\frac{M}{LT}\right]^c \\
&= [M]^a [L]^{a+b+c} [T]^{-a-b-c}
\end{align*}
\]
Physical Sciences 2

Matching exponents, we get $c = 1$ (from $[M]$). From $[T]$ we get $-2 = -b - c$, so $b = 1$. Finally, from $[L]$ we get $a + b - c = 1$, so $a = 1$ also. Thus our final expression is just $F_{\text{drag}} \propto \ell v$. When the viscosity is important, the drag force depends linearly on the length, speed and viscosity.

b) The other expression for drag force depends on the density of the fluid (denoted by $\rho$, the Greek letter rho), but not on the viscosity. Density has SI units of kg·m$^{-3}$. Determine the expression for the magnitude of the drag force that depends on the density of the fluid; again, your answer should be in the form of a proportionality.

The dimensions of density are $[\rho] = [M / L^3]$, so we can just play the same game:

$$\left[ \frac{ML}{T^2} \right] = [L]^a [M]^b [L]^c$$


Once again we get $c = 1$ immediately from $[M]$. Matching $[T]$ gives $b = 2$ directly. Finally, $[L]$ gives $a + b - 3c = 1$, so we get $a = 2$. So the final expression for the drag force is $F_{\text{drag}} \propto \ell^2 v^2 \rho$. When the density is important, the drag force goes like the square of the length and the square of the speed, and is linearly proportional to the density. We’ll see this later in the semester.

The following graph shows the horizontal component of the velocity as a function of time of the chest of a crash-test dummy sitting in the front passenger seat of a car during a 35-mph frontal impact. The four curves on the graph are (from left to right) (i) the velocity of the dashboard, (ii) the velocity of a dummy wearing a seatbelt with special safety devices (pretensioners), (iii) the velocity of a dummy wearing a seatbelt with no special safety devices, and (iv) the velocity of a dummy wearing no seatbelt.

![Graph of velocity vs. time for different seatbelt conditions.](image)


a) Assuming the dummy has a mass of 70 kg, use the graph to estimate the magnitude of the maximum force on the dummy during the collision for each of the three possibilities: the safety-seatbelt, the ordinary seatbelt, and no seatbelt.

Since the graph gives a component of velocity versus time, the slope of each graph at any point indicates the acceleration. Since we are interested in the maximum magnitude of force, we want the slope of the steepest part of each graph. There are various ways of estimating the slope of the graph; perhaps the simplest is to put down a straightedge parallel to the curve and note where it crosses the t-axis (v = 0) and the top of the graph (v = 16 m/s), and then divide $\Delta v$ (which is $-16$ m/s) by whatever $\Delta t$ you find for each curve to get the acceleration (which will be negative).

After you have the accelerations, multiply each acceleration by the mass of the dummy to get the net force on the dummy. (Since we only care about the magnitude, we can discard the minus sign.) Here are my results; your numbers may differ slightly.
Belt with pre-tensioners: slope = 300 m/s², so magnitude of maximum force = $2.1 \times 10^4$ N
Regular belt: slope = 42g = 420 m/s², so magnitude of maximum force = $2.94 \times 10^4$ N
Unbelted: slope = 160g = 1600 m/s², so magnitude of maximum force = $1.12 \times 10^5$ N

b) **Serious tissue injury occurs if the impact force per unit area exceeds about $5 \times 10^5$ N/m².**
Assuming that the forces in a) are distributed over an area of about 500 cm², comment on the severity of the injuries that would likely be sustained in each of the three possible types of impact.

We can merely divide each force by the given area of 500 cm² to get the force per unit area acting on the dummy. 500 cm² is 0.050 m², so:

Belt with pre-tensioners: force per unit area = $4.20 \times 10^5$ N/m²
Regular belt: force per unit area = $5.88 \times 10^5$ N/m²
Unbelted dummy: force per unit area = $2.24 \times 10^6$ N/m²

Even with the safest belt, the force/area is just under the threshold for tissue damage. With the regular belt, it’s over (although this depends on your estimates), and with no seat belt, it’s way beyond the threshold. For a larger person with a larger cross-sectional area, the impact force per area would go down slightly. In contrast, the impact force per area for a smaller person would go up from this estimate.

Either way, the moral of the story is two-fold: one, wear your seatbelt. Two, try not to get into a collision with a fixed barrier at 35 mph, because even seat belts can’t guarantee your safety.

5. **Putting Everything Together: Shoot the monkey (2 pts).**

*In lecture we saw the “Shoot the Monkey” demo. In short a spring-loaded gun is placed on a platform and aimed at a monkey a horizontal distance D away and a vertical distance h off the ground (see figure). Assuming the monkey doesn’t hit the ground first, you will show (by solving the below questions) that it will always get hit. Note: you will need to define some parameters that aren’t given in the drawing to solve this problem.*
a) What is the time it takes the bullet to go a horizontal distance $D$.

In order for the monkey to be hit by the bullet, the bullet and the monkey need to be at the same position in space. The monkey is located some horizontal distance $D$ from the gun and a vertical distance $h$ from the floor. The gun is located on a platform some distance off the floor, which we’re not given; let’s call it $h_g$. Let’s also redefine the monkey’s height to be $h_m$ and the initial velocity of the bullet to be $v_0$ at an angle $\theta$ as shown. First we’ll need to see how long it takes the bullet to traverse the horizontal distance $D$. Then, we’ll use that time to figure out the vertical location of the bullet and compare this to the vertical location of the monkey at that time. If these values are the same, the bullet hits the monkey.

Throughout the problem, let’s pick $x = 0$ to be the barrel of the gun, $y = 0$ to be the floor, and the positive x- and y-axes to point toward the monkey. Let’s also set $t = 0$ to be when the gun fires and the monkey lets go of the branch. The acceleration in the x-direction is zero because the only force acting on the bullet is gravity. The bullet starts at $x_0 = 0$ with an initial x-component of its velocity of $+v_0\cos(\theta)$ and ends up at $x_f = D$. Plugging these in and solving for $t$, we find that

\[
x_f = x_0 + v_{0,x} t + \frac{1}{2} a_x t^2 = 0 + v_0 \cos(\theta) t + 0 = D
\]

or

\[
t = \frac{D}{v_0 \cos(\theta)}.
\]

b) What is the height of the bullet at that time? What is the height of the monkey at that time? Are they the same?

**y-position of bullet**

Now that it’s in the same x-position as the monkey, we want to find the y-position of the bullet. The bullet started at $y = h_g$ and the y-component of its initial velocity is $+v_0\sin(\theta)$. The bullet experiences an acceleration in the y-direction of $-g$. Solving for $y_f$, we get

\[
y_f = y_0 + v_{0,y} t + \frac{1}{2} a_y t^2 = h_g + v_0 \sin(\theta) t + \frac{1}{2} (-g)t^2 = h_g + v_0 \sin(\theta) \frac{D}{v_0 \cos(\theta)} - \frac{g}{2} \frac{D^2}{v_0^2 \cos^2(\theta)}
\]

or

\[
y_f = h_g + D\tan(\theta) - \frac{g}{2} \frac{D^2}{v_0^2 \cos^2(\theta)},
\]

which looks messy, but let’s keep it as is for the time being.
**y-position of monkey**

We need to figure out the monkey’s position after the bullet travels the distance $D$. The monkey starts from rest at a height $y = h_m$, which gives a final position of

$$y_f = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 = h_m + 0 + \frac{1}{2}(-g)t^2 = h_m - \frac{g}{2} \frac{D^2}{v_0^2 \cos^2(\theta)}.$$  

Comparing the equations for the final location of the monkey and the bullet, we see they both have the same final term. The monkey is shot only if its final location is the same as the bullet’s, which is true only if $h_g + D\tan(\theta) = h_m$. From the geometry of the problem (see figure), we can clearly see that this is the case. Therefore, the monkey will *always* get shot if it falls from the branch at the same time the gun is fired.
6. Firing a Mortar Downhill (2 pts)

NOTE: VIDEO TUTORIAL OF THIS PROBLEM ON CANVAS

A mortar crew is positioned near the top of a steep hill. Enemy forces are charging up the hill and it is necessary for the crew to spring into action. Angling the mortar at an angle of $\theta = 52.0^\circ$ (as shown), the crew fires the shell at a muzzle velocity of 258 feet per second.

a) How far down the hill does the shell strike if the hill subtends an angle $\phi = 37.0^\circ$ from the horizontal? (Ignore air friction.)

b) How long will the mortar shell remain in the air?

c) How fast will the shell be traveling when it hits the ground?

The equations of motion are vector equations, and we can therefore express them for individual directions. As seen on the right, the physical properties of the problem have been expressed as vectors and broken down into their components, with blue vectors corresponding to x-components and red vectors corresponding to y-components. (The component for $v_0\cos(\theta)$ is not shown due to lack of space, but its blue arrow representation is displayed.)

We can now cast the displacement equation of motion

$$\vec{d} = \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

to express motion along the x-direction and y-direction.

$$\vec{d}_x = v_{0,x} t + \frac{1}{2} a_x t^2$$

$$\vec{d}_y = v_{0,y} t + \frac{1}{2} a_y t^2$$

Examining the figure we can substitute expressions for many of the physical properties appearing in these two equations, carefully noting the direction of the vector. Note that $a_x = 0$ and $a_y = g$ (free fall).
Physical Sciences 2

\[ d \cos(\phi) = v_0 \cos(\theta) t \]

\[ -d \sin(\phi) = v_0 \sin(\theta) t + \frac{1}{2} (-g) t^2 \]

Notice the signs (plus or minus) on the equation for the y-direction; the displacement \(d\sin(\phi)\) and acceleration \(g\) are assigned negative directions since both vectors point downward.

We can now solve for the time of flight \(t\) in the first equation

\[ t = \frac{d \cos(\phi)}{v_0 \cos(\theta)} \]

and substitute the result for \(t\) in the second equation:

\[ -d \sin(\phi) = \frac{v_0 t \cos(\theta) \cos(\phi)}{v_0 \cos(\theta)} - \frac{g d^2 \cos^2(\phi)}{2 v_0^2 \cos^2(\theta)} \]

Solving for the distance \(d\) (that is, the distance down the hill at which the mortar shell lands) we get

\[ d = \frac{2 \cos(\theta) v_0^2}{g \cos^2(\phi)} \left( \cos(\theta) \sin(\phi) + \sin(\theta) \cos(\phi) \right) \]

We can now substitute the numerical values stated in the problem. First, we should express the muzzle velocity \(v_0\) in terms of SI units.

\[ v_0 = 258 \text{ ft/s} = \left(258 \text{ ft/s} \right) \left( \frac{0.304800 \text{ m}}{1 \text{ ft}} \right) = 78.6384 \text{ m/s} \]

We can now calculate the distance \(d\) the shell lands down the hill.

\[ d = \frac{2 \cos(52.0^\circ) \left(78.6384 \text{ m/s} \right)^2 \left( \cos(52.0^\circ) \sin(37.0^\circ) + \sin(52.0^\circ) \cos(37.0^\circ) \right)}{\left(9.81 \text{ m/s}^2 \right) \cos^2(52.0^\circ)} = 1220 \text{ m} \]

We can find the time of flight from an equation we derived earlier.

\[ t = \frac{d \cos(\phi)}{v_0 \cos(\theta)} = \frac{\left(1220 \text{ m} \right) \cos(37.0^\circ)}{\left(78.6384 \text{ m/s} \right) \cos(52.0^\circ)} = 20.1 \text{ s} \]

To find the final speed of the shell, we note that the horizontal component of the velocity will remain constant since there is no acceleration in the x-direction.
Physical Sciences 2

\[ v_{1,x} = v_{0,x} = v_0 \cos(\phi) = \left( 78.6384 \text{ m/s} \right) \cos(52.0^\circ) = 48.416 \text{ m/s} \]

The final velocity in the y-direction is given by

\[ v_{1,y} = v_0 \sin(\theta) - gt = \left( 78.6384 \text{ m/s} \right) \sin(52.0^\circ) - \left( 9.81 \text{ m/s}^2 \right) \left( 20.1 \text{ s} \right) = -134.93 \text{ m/s} \]

where the negative sign indicates that the shell is dropping in height. Now that we have the horizontal and vertical components of the final velocity vector, the final speed is given by the Pythagorean theorem.

\[ v_1 = \sqrt{v_{1,x}^2 + v_{1,y}^2} = \sqrt{(48.416 \text{ m/s})^2 + (-134.93 \text{ m/s})^2} = 143 \text{ m/s} \]