Volumes in dimension 2 are obtained by counting lattice points.

**Dimension 3**

Can we count lattice points to find the volume of polyhedra?

**Example (Reeve tetrahedra)**

Consider the tetrahedron with lattice points \((0,0,0), (1,0,0), (0,0,1), \) and \((1,1,1)\).

Volume: \(\frac{1}{6} \text{base height} / 3\)

\(I = 0, \quad Z = 4. \quad \text{This is not possible!}\)

Work around: a solid like Reeve's tetrahedron would capture many lattice points if we would scale it (equally) in all directions.

**Lattice-point enumerator**

Let \(P\) be a polytope that is the convex hull of the points \(s_1, \ldots, s_m\). Let \(t s_i\) be the dilate of the point \(s_i\), i.e.

\([t v_1, \ldots, v_n] = [tv_1, tv_2, \ldots, tv_n]\).

The \(t\)th dilate of \(P\) is the convex hull of \(ts_1, \ldots, ts_m\), written \(tP\).

**Example**

![Diagram of lattice points and dilates](image_url)
The lattice-point enumerator of $P$ counts the integral points in $tP$, including the boundary:

$$L_p(t) = \#(tP \cap \mathbb{Z}^d) = \#(P \cap \frac{1}{t} \mathbb{Z}^d)$$

This is also called the discrete volume of $P$.

Example

Consider $P$ to be the convex hull of $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.

The lattice-point enumerator of the square is

$$L_p(t) = (t+1)^2$$

The area is $t^2$, and $L_p(t) < t^2 < L_p(t)$

Example

Consider $P$ to be the convex hull of $(0,0)$, $(1,0)$ and $(0,1)$.

The lattice point of the right triangle is $L_p(t) = \frac{(t+1)(t+2)}{2}$.
Example

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_{\Delta_{1}^{(h)}} )</td>
<td>7</td>
<td>20</td>
<td>40</td>
<td>67</td>
<td>( \frac{3}{2} t^2 + \frac{5}{2} t + 1 )</td>
</tr>
<tr>
<td>( L_{\Delta_{0}^{(h)}} )</td>
<td>2</td>
<td>10</td>
<td>25</td>
<td>47</td>
<td>( \frac{3}{2} t^2 + \frac{5}{2} t + 1 )</td>
</tr>
</tbody>
</table>

The lattice-point enumerator of \( P \) is \( \frac{3}{2} t^2 + \frac{5}{2} t + 1 \), and the area of \( P \) is \( \frac{3}{2} t^2 \).

Observations

- \( L_{p_{1}}(t) \) and \( L_{p_{0}}(t) \) are polynomials of degree 2 when \( \dim(P) = 2 \).
- \( L_{p_{0}}(t) \) < \( \text{area}(P) \) < \( L_{p_{1}}(t) \).

In dimension 2, Pick's theorem implies

\[
\text{area}(P) = \frac{L_{p_{0}}(t) + L_{p_{1}}(t) - 1}{2}
\]

Proof:

\[
\text{area}(P) = \frac{1}{2} \text{I}(P) + \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) - 1
\]

\[
= L_{p_{0}}(t) + \frac{1}{2} \left( L_{p_{1}}(t) - L_{p_{0}}(t) \right) - 1
\]

\[
= \frac{L_{p_{0}}(t) + L_{p_{1}}(t) - 1}{2}
\]
The lattice point enumerator works in higher dimensions.

**Example**: $d$-cube.

Consider $P$ to be the $d$-cube, i.e., the convex hull of the points in $\{0,1\}^d$.

- $L_p(t) = (t+1)^d$
- $L_{p^o}(t) = (t-1)^d$

**Observations**: $L_{p^o}(t) < \text{Volume } (P) < L_p(t)$.

- It is not true that the volume is $L_p(t) + L_{p^o}(t) - 1$.

**Example**: $d$-simplex.

Consider $P$ to be the $d$-simplex, i.e., the convex hull of the origin and the elementary vectors $\{e_1, \ldots, e_d | e_i = (0,0,\ldots,0,1,0,\ldots,0)\}$.

In dimension 2, the 2-simplex is a right triangle with sides parallel to the axes.

To count the number of integral points in the $d$-simplex, we reformulate, using the following inequality: If $V=(v_1, v_2, \ldots, v_d)$ is in the $t$-th dilate of the simplex (including the boundary), then

$$v_1 + \ldots + v_d \leq t,$$

with $v_1, \ldots, v_d \in \mathbb{Z}_{\geq 0}$

which is the same as

$$v_1 + \ldots + v_d + v_{d+1} = t,$$

with $v_{d+1} \in \mathbb{Z}_{\geq 0}$

"slack"
To solve this equation, we use generating functions. Let \( z \) be a formal variable.

Then, the number of solutions to \( v_1 + \ldots + v_d + v_{d+1} - t = 0 \) is the constant term of

\[
\sum_{v_1, \ldots, v_{d+1} \geq 0} z^{v_1 + v_2 + \ldots + v_d + v_{d+1} - t} = \sum_{v_1, \ldots, v_{d+1} \geq 0} z^{v_1} z^{v_2} z^{v_3} \ldots z^{v_d} z^{v_{d+1}} z^{-t}
\]

\[
= \frac{z^{v_1}}{1-z} \frac{z^{v_2}}{1-z} \ldots \frac{z^{v_{d+1}}}{1-z} \frac{1}{1-z} \frac{1}{z^{-t}}
\]

\[
= \left( \frac{1}{1-z} \right)^{d+1} \frac{1}{z^t}
\]

To find the constant term of that polynomial, we expand it. We will actually find the coefficient in front of \( z^t \) of

\[
\left( \frac{1}{1-z} \right)^{d+1}
\]

It is known that

\[
\frac{z^d}{(1-z)^{d+1}} = \sum_{k \geq d} \binom{k}{d} z^k.
\]

(Generating function for binomial coefficients)

Hence,

\[
\frac{1}{(1-z)^{d+1}} = \sum_{k \geq d} \binom{k}{d} z^{k-d}
\]

\[
= \sum_{l \geq 0} \binom{d+l}{d} z^l, \quad (\text{by setting } l = k-d)
\]

and the coefficient of \( z^t \) is \( \binom{t+d}{d} \).

Hence, \( L_p(t) = \binom{t+d}{d} \) when \( P \) is the \( d \)-simplex.
Ehrhart series

It is possible to keep track of all the information about a family of polytopes at once. It is done through Ehrhart series:

**Definition**

The Ehrhart series of a polytope $P$ is given by

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t.$$  

**Example**

The Ehrhart polynomial for the $d$-simplex is

$$1 + \sum_{t \geq 1} \binom{t+d}{d} z^t = \sum_{t \geq 0} \binom{t+d}{d} z^t = \frac{1}{1-z}.$$  

The Ehrhart polynomial for the $(2$-dimensional$)$ square is

$$1 + \sum_{t \geq 1} (t+1)^2 z^t = \sum_{t \geq 0} (t+1)^2 z^t$$

$$= \frac{d}{dz} z \sum_{t \geq 0} (t+1) z^t$$

$$= \frac{d}{dz} z \frac{d}{dz} \frac{1}{1-z}$$

$$= \frac{z}{(1-z)^2}.$$  

Hence,

$$Ehr_{\Box}(z) = \frac{z+1}{(1-z)^3}.$$
**Theorem (Ehrhart)**

Let \( P \) be a \( d \)-dimensional integral polytope. Then,

- \( L_p(t) \) is a polynomial of degree \( d \)
- \( \text{Ehr}_p(z) = g(z) \frac{1}{(1-z)^d} \), where \( g \) is a polynomial of degree at most \( d \) with \( g(1) \neq 0 \).

**Theorem (Stanley)**

Let \( P \) be a \( d \)-dimensional lattice polytope. Then,

\[
\text{Ehr}_p(z) = h_d z^d + h_{d-1} z^{d-1} + \ldots + h_0 z^0 \frac{1}{(1-z)^d},
\]

Interesting combinatorially.

with \( h_0, h_1, \ldots, h_d \) non-negative integers.

**Reference:** [CCDJ, 62,3]