Math 108 - Geometric combinatorics Nadia Lafrenière

Volumes in higher dimension: Ehrhart theory 1/27/2023

Volumes in dimension 2 are obtained by counting lattice points.

## Dimension 3

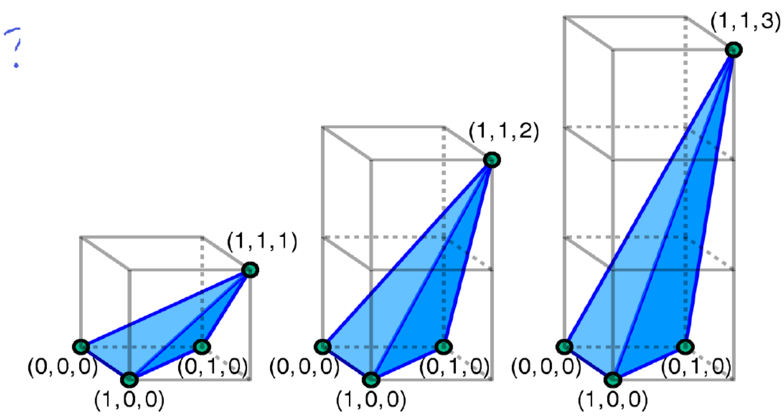
Can we count lattice points to find the volume of polyhedra?

Example (Reeve tetrahedra)

Consider the tetrahedron with lattice points $(0,1,0)$, $(1,0,0)$, $(0,0,0)$ and $(1,1,n)$.

volume: $n/6$ (base · height / 3)

$I = 0$, $B = 4$. This is not possible!




Picture: CMG Lee on Wikipedia

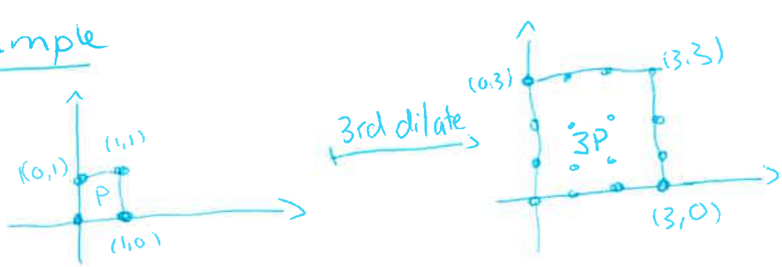
Work around: a solid like Reeve's tetrahedron would capture many lattice points if we would scale it (equally) in all directions.

## Lattice-point enumerator

Let $P$ be a polytope that is the convex hull of the points $s_1, \dots, s_m$. Let $ts_i$ be the dilate of the point $s_i$, i.e. $t(v_1, \dots, v_n) = (tv_1, tv_2, \dots, tv_n)$.

The $t$-th dilate of $P$ is the convex hull of $ts_1, \dots, ts_m$, written $tP$.

Example



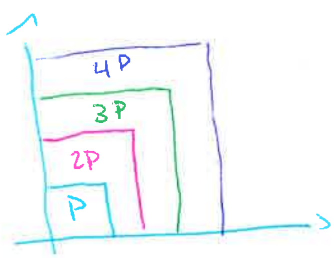
The lattice-point enumerator of $P$ counts the integral points in $tP$, including the boundary:

$$L_P(t) := \#(tP \cap \mathbb{Z}^d) = \#(P \cap \underbrace{\tfrac{1}{t}\mathbb{Z}^d}_{\text{"shrinking" of the grid.}})$$

This is also called the discrete volume of $P$.

## Example

Consider $P$ to be the convex hull of $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.




| $t$ | 1 | 2 | 3 | 4 | | $t$ |
|---|---|---|---|---|---|---|
| $L_{\square}(t)$ | 4 | 9 | 16 | 25 | ... | $(t+1)^2$ |
| $L_{\underbrace{\square^\circ}_{\text{interior of } P}}(t)$ | 0 | 1 | 4 | 9 | ... | $(t-1)^2$ |

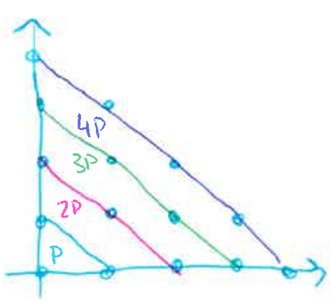
The lattice-point enumerator of the square is

$$L_P(t) = (t+1)^2$$

The area is $t^2$, and $L_{P^\circ}(t) < t^2 < L_P(t)$

## Example

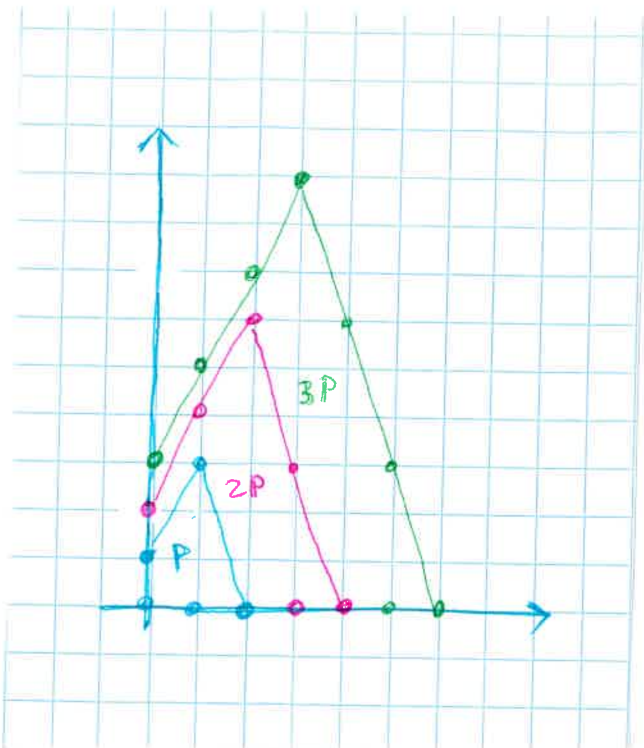
Consider $P$ to be the convex hull of $(0,0)$, $(1,0)$ and $(0,1)$




| $t$ | 1 | 2 | 3 | 4 | ... | $t$ |
|---|---|---|---|---|---|---|
| $L_\Delta(t)$ | 3 | 6 | 10 | 15 | ... | $\binom{t+2}{2} = \frac{(t+1)(t+2)}{2}$ |
| $L_{\Delta^\circ}(t)$ | 0 | 0 | 1 | 3 | ... | $\binom{t-1}{2} = \frac{(t-1)(t-2)}{2}$ |

The lattice point of the right triangle is $L_P(t) = \frac{(t+1)(t+2)}{2}$

## Example




| $t$ | 1 | 2 | 3 | 4 | $t$ |
|---|---|---|---|---|---|
| $L_{\triangle}(t)$ | 7 | 20 | 40 | 67 | $\cdots \frac{7}{2}t^2+\frac{5}{2}t+1$ |
| $L_{\triangle^\circ}(t)$ | 2 | 10 | 25 | 47 | $\cdots \frac{7}{2}t^2-\frac{5}{2}t+1$ |

The lattice-point enumerator of $P$ is $L_P(t) = \frac{7}{2}t^2 + \frac{5}{2}t + 1$, and the area of $P$ is $\frac{7}{2}t^2$.

## Observations

- $L_P(t)$ and $L_{P^\circ}(t)$ are polynomials of degree 2 when $\dim(P) = 2$
- $L_{P^\circ}(t) < \text{area}(tP) < L_P(t)$.

In dimension 2, Pick's theorem implies

$$\text{area}(tP) = \frac{L_{P^\circ}(t) + L_P(t) - 1}{2}.$$

Proof:

$$\begin{aligned}
\text{area}(tP) &= I(tP) + \frac{1}{2}B(tP) - 1 \\
&= L_{P^\circ}(t) + \frac{1}{2}\left(L_P(t) - L_{P^\circ}(t)\right) - 1 \\
&= \frac{L_{P^\circ}(t) + L_P(t)}{2} - 1.
\end{aligned}$$

The lattice point enumerator works in higher dimensions. 

Example d-cube

Consider $P$ to be the d-cube, i.e. the convex hull of the points in $\{0,1\}^d$.

$$L_P(t) = (t+1)^d \qquad \text{Volume} = t^d$$

$$L_{P^\circ}(t) = (t-1)^d$$

Observations:
- $L_{P^\circ}(t) < \text{Volume}(P) < L_P(t)$
- It is not true that the volume is $\frac{L_P(t) + L_{P^\circ}(t)}{2} - 1$.

Example: d-simplex

Consider $P$ to be the d-simplex, i.e. the convex hull of the origin and the elementary vectors $\{e_1, \dots, e_d \mid e_i = (0,0,\dots,\underset{\underset{i\text{th}}{\uparrow}}{1},\dots,0)\}$

In dimension 2, the 2-simplex is a right triangle with sides parallel to the axes.

To count the number of integral points in the d-simplex, we reformulate, using the following inequality: If $v = (v_1, v_2, \dots, v_d)$ is in the t-th dilate of the simplex (including the boundary), then

$$v_1 + \dots + v_d \leq t, \quad \text{with } v_1, \dots, v_d \in \mathbb{Z}_{\geq 0}$$

which is the same as

$$v_1 + \dots + v_d + \underbrace{v_{d+1}}_{\text{"slack"}} = t, \quad \text{with } v_{d+1} \in \mathbb{Z}_{\geq 0}$$

To solve this equation, we use generating functions.

Let $z$ be a formal variable.

Then, the number of solutions to $v_1 + \dots + v_d + v_{d+1} - t = 0$ is the constant term of

$$\sum_{v_1, \dots, v_{d+1} \geq 0} z^{v_1 + v_2 + \dots + v_d + v_{d+1} - t} = \sum_{v_1, \dots, v_{d+1} \geq 0} z^{v_1} z^{v_2} z^{v_3} \dots z^{v_d} z^{v_{d+1}} z^{-t}$$

$$= \underbrace{\sum_{v_1 \geq 0} z^{v_1}}_{\frac{1}{1-z}} \sum_{v_2 \geq 0} z^{v_2} \dots \sum_{v_{d+1} \geq 0} z^{v_{d+1}} \cdot z^{-t}$$

$$= \left(\frac{1}{1-z}\right)^{d+1} \cdot \frac{1}{z^t}$$

To find the constant term of that polynomial, we expand it. We will actually find the coefficient in front of $z^t$ of $\left(\frac{1}{1-z}\right)^{d+1}$. It is known that

$$\frac{z^d}{(1-z)^{d+1}} = \sum_{k \geq d} \binom{k}{d} z^k.$$ (Generating function for binomial coefficients)

Hence,

$$\frac{1}{(1-z)^{d+1}} = \sum_{k \geq d} \binom{k}{d} z^{k-d}$$

$$= \sum_{l \geq 0} \binom{l+d}{d} z^l,$$ (by setting $l = k-d$)

and the coefficient of $z^t$ is $\binom{t+d}{d}$.

Hence, $L_P(t) = \binom{t+d}{d}$ when $P$ is the $d$-simplex.

# Ehrhart series

It is possible to keep track of all the information about a family of polytopes at once. It is done through Ehrhart series:

## Definition

The Ehrhart series of a polytope $P$ is given by

$$\mathrm{Ehr}_P(z) = 1 + \sum_{t\geq 1} L_P(t)\, z^t.$$

## Example

- The Ehrhart polynomial for the $d$-simplex is

$$1+\sum_{t\geq 1}\binom{t+d}{d} z^t = \sum_{t\geq 0}\binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}}$$

← For free, using computations from last page.

- The Ehrhart polynomial for the (2-dimensional) square is

$$1+\sum_{t\geq 1}(t+1)^2 z^t = \sum_{t\geq 0}(t+1)^2 z^t$$

$$= \frac{d}{dz}\sum_{t\geq 0}(t+1)\, z^{t+1}$$

$$= \frac{d}{dz}\, z \sum_{t\geq 0}(t+1)\, z^t$$

$$= \frac{d}{dz}\, z \frac{d}{dz}\sum_{t\geq 0} z^{t+1}$$

$$= \frac{d}{dz}\, z\, \frac{d}{dz}\, \frac{z}{1-z}$$

$$= \frac{d}{dz}\, \frac{z}{(1-z)^2}$$

$$= \dots = \frac{z+1}{(1-z)^3}$$

Hence,

$$\mathrm{Ehr}_{\square}(z) = \frac{z+1}{(1-z)^3} = \frac{z+1}{(1-z)^{d+1}}.$$

**Theorem** (Ehrhart)

Let $P$ be a $d$-dimensional integral polytope. Then,

- $L_P(t)$ is a polynomial of degree $d$
- $Ehr_P(z) = \frac{g(z)}{(1-z)^{d+1}}$, where $g$ is a polynomial of degree at most $d$ with $g(1) \neq 0$.

**Theorem** (Stanley)

Let $P$ be a $d$-dimensional lattice polytope. Then,

$$Ehr_P(z) = \frac{h_d^* z^d + h_{d-1}^* z^{d} + \dots + h_0^*}{(1-z)^{d+1}},$$ ← $h^*$-polynomial. Interesting combinatorially.

with $h_0^*, h_1^*, \dots, h_d^*$ non-negative integers.

**Reference**: [CCD], §2,3.