Linear operators:

A <u>linear operator</u> on V (aka <u>endomorphism</u> of V) is a linear map $\varphi: V \to V$.

<u>Notation</u>: $End(V) = Hom(V,V)$.

* When using a basis to express $\varphi \in Hom(V,V)$ by a (square) matrix, we want to use the same basis on each side: $A = \mathcal{M}(\varphi, (e_i), (e_i))$, transforms by $P^{-1}AP$.

* <u>Observe</u>: if $\dim V < \infty$, $\varphi: V \to V$ is injective $\Leftrightarrow$ surjective $\Leftrightarrow$ isom. $\Leftrightarrow$ rank $\varphi = \dim V$. (then say φ is invertible, $\varphi^{-1}: V \to V$ the inverse linear operator).

* New thing: we can <u>compose</u> linear operators with each other $\varphi\psi = \varphi \circ \psi : V \to V$ or with themselves, $\varphi^n = \underbrace{\varphi \circ \dots \circ \varphi}_{n \text{ times}}$, or even apply polynomials: $p = \sum a_n x^n \rightsquigarrow p(\varphi) = \sum a_n \varphi^n : V \to V$.

  $Hom(V,V)$ is a (noncommutative) <u>ring</u>.

* Given vector spaces V_1, V_2 and linear operators $\varphi_i : V_i \to V_i$, we can define
  $\varphi = \varphi_1 \oplus \varphi_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ operator on $V = V_1 \oplus V_2$.

  The operator φ leaves the subspaces $V_1, V_2 \subset V$ <u>invariant</u>: $\varphi(V_i) \subset V_i$, and working in a basis of V st. $e_1 \dots e_m \in V_1$, $e_{m+1} \dots e_n \in V_2$, the matrix of φ is <u>block diagonal</u>: $\left(\begin{array}{c|c} \varphi_1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right)$. Conversely, if $V = V_1 \oplus V_2$ and $\varphi(V_i) \subset V_i$ then φ is of this form.

  More generally, if we only assume $\varphi: V \to V$ and $V_1 \subset V$ is invariant ($\varphi(V_1) \subset V_1$) but not necess. V_2, then the matrix of φ would be <u>block triangular</u>: $\left(\begin{array}{c|c} \varphi_{/V_1} & * \\ \hline 0 & * \end{array}\right)$

  <u>So</u>: a typical way to study $\varphi: V \to V$ is to look for invariant subspaces.

* If $U \subset V$ is invariant and $\dim U = 1$ (so: $U = k \cdot v$ for some $v \in V$), then necessarily $\varphi(v) = \lambda v$ for some $\lambda \in k$.

<u>Def</u>: An <u>eigenvector</u> of $\varphi: V \to V$ is a vector $v \in V, v \neq 0$, st. $\varphi(v) = \lambda v$ for some $\lambda \in k$. λ is called the <u>eigenvalue</u> corresponding to v.

* If we can find a basis of V consisting of eigenvectors of φ, then we have diagonalized φ, i.e. found a basis where its matrix is diagonal

$$\varphi(v_i) = \lambda_i v_i \qquad \begin{array}{c} \quad v_1 \;\cdots\; v_n \\ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{array}$$

This is the best outcome, but not always possible!

Ex: $V=\mathbb{R}^2$, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has eigenvectors $\begin{matrix}(1,0)\\(0,1)\end{matrix}$ (or any multiples) with eigenvalues $\begin{matrix}\lambda\\ \mu.\end{matrix}$

However $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one eigenvector $(1,0)$ with eigenvalue 1, (up to scaling!) NOT diagonalizable.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ doesn't even have any eigenvectors (see HW4 Problem 1).

Prop: Eigenvectors of $\varphi: V \to V$ with distinct eigenvalues are linearly independent.

Pf: Assume $v_1 \dots v_\ell$ are eigenvectors with $\varphi(v_i) = \lambda_i v_i$, λ_i all distinct.
Assume there is a linear relation $a_1 v_1 + \dots + a_\ell v_\ell = 0$ with a_i not all zero.
and this has the fewest (≥ 2) possible nonzero a_i of any such relation

Then $\varphi(\sum a_i v_i) = \sum a_i \varphi(v_i) = \sum a_i \lambda_i v_i = 0$ another linear relation!

$$a_1 \lambda_1 v_1 + \dots + a_\ell \lambda_\ell v_\ell = 0.$$

Fix i st. $a_i \neq 0$, and subtract:

$$a_1(\lambda_1 - \lambda_i) v_1 + \dots + a_\ell(\lambda_\ell - \lambda_i) v_\ell = 0$$

$\to$ linear relation where coefficient of v_i is now zero, but all other nonzero coefficients (at least one) remain nonzero (since $\lambda_j - \lambda_i \neq 0$).
Contradicts minimality assumption. $\square$

Corollary: The number of distinct eigenvalues of $\varphi \in \mathrm{Hom}(V,V)$ is at most $n = \dim V$, and if equality holds then φ is diagonalizable.

Digression: Def: A field k is algebraically closed if every nonconstant polynomial $p \in k[x]$ has a root in k, ie. $\exists \alpha \in k$ st. $p(\alpha) = 0$.

If so, then by division algorithm for polynomials, can write $p = (x-\alpha) q$, and repeating, we get $p = c(x-\alpha_1)\dots(x-\alpha_d)$. ($d = \deg p$, $\alpha_i \in k$).

* Fundamental theorem of algebra: $\mathbb{C}$ is algebraically closed.
(proof is not pure algebra; we'll discuss it in Math 556).

* If k is not algebraically closed then there exists an alg. closed field $\bar{k} \supset k$, constructed from k by adjoining roots of polynomials $\in k[x]$.

Eg. $\bar{\mathbb{R}} = \mathbb{C}$, whereas $\bar{\mathbb{Q}} = \{$all roots of polynomial eq^{ns} with $\mathbb{Q}$-coeffts$\} \subset \mathbb{C}$

(fact: polynomials in $\bar{\mathbb{Q}}[x]$ have roots in $\bar{\mathbb{Q}}$)

<u>Lemma:</u> If k is algebraically closed, V a finite dimensional vector space over k, then any linear operator $\varphi : V \to V$ has an eigenvector, ie. $\exists v \in V - \{0\}$, $\exists \lambda \in k$ st. $\varphi(v) = \lambda v$.

<u>Proof:</u> Let $n = \dim V$, and take any nonzero vector $v \in V$. Then $\underbrace{v, \varphi(v), \dots, \varphi^n(v)}_{n+1 \text{ vectors}}$ must be linearly dependent.

So $\exists a_0, \dots, a_n \in k$ (not all zero) st. $a_0 v + a_1 \varphi(v) + \dots + a_n \varphi^n(v) = 0$.

Since k is algebraically closed, we can factor the polynomial $\sum a_i x^i$,

hence $a_0 + a_1 \varphi + \dots + a_n \varphi^n = c(\varphi - \lambda_1) \dots (\varphi - \lambda_d)$, $c \neq 0$, $\lambda_i \in k$.

(!! the product here is composition of operators, but this is legit !!)

Now, $(\varphi - \lambda_1) \dots (\varphi - \lambda_d) : V \to V$ has a nontrivial kernel $(\ni v)$, which implies that at least one of $\varphi - \lambda_i$ is not an isomorphism,

hence $\exists i \in \{1..d\}$ and $w \in V - \{0\}$ st. $w \in Ker(\varphi - \lambda_i)$, ie. $\varphi(w) = \lambda_i w$. $\square$

<u>Corollary:</u> Given $\varphi : V \to V$ over an algebraically closed field k, there exists a basis $(v_1, \dots, v_n)$ of V in which the matrix of φ is upper-triangular.

(ie. each subspace $V_k = span(v_1 \dots v_k) \subset V$ is invariant)

<u>Proof:</u> • Induction on $\dim V$: If $\dim V = 1$, then any nonzero vector v_1 gets mapped to a multiple of itself $\checkmark$. (any 1×1 matrix is triangular)

• Assume result true for dim. $\leq n-1$, and consider $\varphi : V \to V$ with $\dim V = n$. By lemma, φ has at least one eigenvalue $\lambda \in k$. Let $U = Im(\varphi - \lambda)$. Since $\varphi - \lambda$ has nontrivial kernel (= eigenvectors for λ), $\dim U < \dim V$. Moreover, we claim U is an invariant subspace for φ.

Indeed: if $u = (\varphi - \lambda) v \in Im(\varphi - \lambda) = U$, then

$$\varphi(u) = \varphi(\varphi - \lambda) v = (\varphi - \lambda)\varphi(v) \in Im(\varphi - \lambda) = U.$$

Now, by induction, $\varphi_{|U} \in Hom(U, U) \rightsquigarrow \exists$ basis $u_1, \dots, u_m$ of U in which $\varphi_{|U}$ is upper-triangular. $(\varphi(u_i) \in span(u_1 \dots u_i))$

Complete to a basis $(u_1 \ldots u_m, v_1 \ldots v_k)$ of V. Then:

- $\varphi(u_i) \in \mathrm{span}(u_1 \ldots u_i)$ ✓
- $\varphi(v_i) = \underbrace{(\varphi-\lambda)v_i}_{\in U} + \lambda v_i \in \mathrm{span}(u_1 \ldots u_m, v_i)$. ✓

$$\Rightarrow \mathcal{M}(\varphi) = \begin{array}{c} \begin{array}{cc} U & \end{array} \\ \left(\begin{array}{c|c} \mathcal{M}(\varphi_{|U}) & * \\ \hline 0 & \begin{matrix} \lambda & 0 \\ 0 & \ddots \lambda \end{matrix} \end{array}\right) \end{array}$$

□

★ Rmk: there's another proof that is easier to discover but harder to follow: again by induction, but now start from $V_0 = k \cdot v_0$ where v_0 is an eigenvector of φ, and let $U = V/V_0$, $q: V \to U$ quotient.

Using $\varphi(V_0) \subset V_0$, $\exists\, \bar\varphi : U \to U$ st.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & V \arrow[d, "q"] \\ U \arrow[r, "\bar\varphi"] & U \end{tikzcd}$$

commutes

(because: $(q \circ \varphi)_{|V_0} = 0$ so $q \circ \varphi : V \to U$ factors through $V/V_0 = U$).

$$\mathcal{M}(\varphi) = \begin{array}{c} \begin{array}{cc} v_0 & \end{array} \\ \left(\begin{array}{c|c} \lambda & * \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \mathcal{M}(\bar\varphi) \end{array}\right) \end{array}$$

By induction hypothesis, $\exists$ basis $u_1, \ldots u_{n-1}$ of U st. $\bar\varphi(u_i) \in \mathrm{span}(u_1 \ldots u_i)$.

Let $v_i \in V$ such that $q(v_i) = u_i$. Then $q(\varphi(v_i)) \in \mathrm{span}(u_1 \ldots u_i)$

(Note: $(v_0 \ldots v_{n-1})$ basis of V). $\Rightarrow \varphi(v_i) \in \mathrm{span}(v_0, v_1 \ldots v_i)$. □

Now suppose we have $\varphi : V \to V$ and a basis $(v_1, \ldots, v_n)$ of V st $\mathcal{M}(\varphi) = A$ is upper-triangular, ie. each $V_i = \mathrm{span}(v_1, \ldots, v_i)$ is an invariant subspace of φ.

Denote by $\lambda_i = a_{ii}$ the diagonal entries of A.

Lemma: ‖ φ is invertible iff all the diagonal entries of A are nonzero.

PF:
- if all λ_i are nonzero then φ is surjective (hence isom.) since
  $\varphi(v_1) = \lambda_1 v_1$, $\lambda_1 \neq 0$ so $v_1 \in \mathrm{Im}\,\varphi$
  $\varphi(v_2) = \lambda_2 v_2 + a_{12} v_1$, $\lambda_2 \neq 0$ so $v_2 = \frac{1}{\lambda_2}(\varphi(v_2) - a_{12} v_1) \in \mathrm{Im}\,\varphi$
  etc. $\Rightarrow v_i \in \mathrm{Im}\,\varphi \ \forall i$.
- if $\lambda_i = 0$ then $\varphi(V_i) \subset V_{i-1}$ so $\varphi_{|V_i}$ has nontrivial kernel (since $\mathrm{rk}\, \varphi_{|V_i} \le \dim V_{i-1} < \dim V_i$), hence $\ker \varphi \neq 0$, not invertible. □

Corollary: ‖ The following are equivalent:
(1) λ is an eigenvalue of φ
(2) $\varphi - \lambda$ is not invertible
(3) $\lambda = \lambda_i$ for some diagonal entry of the upper-triangular matrix A representing φ.

((1)$\Leftrightarrow$(2) since eigenvectors $= \ker(\varphi - \lambda)$, and (2)$\Leftrightarrow$(3) by applying the lemma to $\varphi - \lambda$ and matrix $A - \lambda I$)