than those associated with the Fermi surface.

(c) An even more subtle state where there is a sharp Fermi surface with Landau quasiparticles but there are extra zero energy states when system is put on a torus but not with free boundary conditions.

In practice in many metals various techniques can be used to determine the shape of the Fermi surface.

E.g.: de Haas-van Alphen or Schubnikov-de Haas oscillations.

These involve oscillation of some physical quantity as a function of $\nu \theta$ and is ultimately due to Landau quantization of the orbits.

\[
\frac{M}{B} \uparrow
\]

To see these need to go to $B \ll B_c$

Temperatures low enough that $k_B T \ll \hbar \omega_c$ with

\[
\frac{\nu \theta}{B} \quad \omega_c = \text{cyclotron frequency} = \frac{eB}{mc}
\]

Note: (The amplitude is suppressed by a factor $e^{-2\pi^2 \frac{k_B T}{\hbar \omega_c}}$).

Also need very clean samples: the amplitude has an additional "Dingle factor" $e^{-2\pi^2 \frac{T}{\hbar \omega_c}}$ with $T =$ impurity scattering rate.
\[ \Rightarrow \text{Need } \frac{W}{T} \to \text{that quasiparticles can complete a full cyclotron orbit.} \]

Note that the effective mass \( m^* \) of the quasiparticles enters \( \frac{W}{T} \).

By carefully studying the amplitude of the oscillations as a function of \( B \) and \( T \), one can extract \( m^* \) and \( T \).

The period of the oscillations itself contains information about the size and shape of the Fermi surface (see, e.g., Ashcroft & Mermin).

---

**Green's functions** A very useful characterization of a many-body state is in terms of the "Green's function" for various physical operators. In the context of interacting fermion systems, these are very useful for several reasons:

(a) Provide a way to formalize the question of whether electron-like quasiparticles exist or not.

(b) Contain fundamental measurable information about the existence and properties of quasiparticles (or the nature of the excitation spectrum if there are no quasiparticles).

(c) Basis for a perturbative treatment of the interacting fermion problem.
To motivate the expected structure & definition of Greens functions consider the quantum state obtained by acting with the electron creation operator on the ground state $c_k^+ |\psi_{gd}\rangle$. If the added electron has momentum $\vec{p}$ so that the resulting state is a low energy state of the $(N+1)$-electron system then expect that this state will have some non-zero overlap with the (nearly) exact quasiparticle eigenstate i.e.

$$c_k^+ |\psi_{gd}\rangle \approx \sqrt{Z} |\varphi_k\rangle + \cdots$$

Here $|\varphi_k\rangle$ is the quasiparticle state, $0 < Z \leq 1$ is the overlap, and $\cdots$ refers to states where many quasiparticles/holes are excited.

Consider now the following probe of the system. Suppose that at time $t = 0$ we add an electron at momentum $\vec{p}$, then wait a time $t$ and see if the electron is still there at $\vec{p}$. This may be probed by the expectation value

$$\theta(t) \langle c_k(t) c_k^+(0) \rangle \equiv i \langle \mathbf{g}(\vec{p}, t) \rangle$$

To evaluate LHS, insert a complete set of states in the middle
\[ i g(t) = c(t) \sum_n \langle gd | \xi_k(t) | n \rangle \langle n | \xi_k^* | gd \rangle \]

Note that \( |n\rangle \) has 1 extra particle.

Of the states \( |n\rangle \) for which the matrix elements do not vanish, expect that the single q/p state has weight of \( o(1) \) while other states each have infinitesimal weight.

\[ i g(\vec{k}t) \propto c(t) \left[ Z e^{-i\xi_k t} + \ldots \right] \]

In general however we must allow for the fact that the q/p states are not exact eigenstates but decay at some rate \( \gamma \ll \xi_k \). Thus we must modify \( e^{-i\xi_k t} \) to \( e^{-i\xi_k t} e^{-\gamma t} \) with \( \gamma \ll \xi_k \). Then

\[ i g(\vec{k}t) \propto c(t) \left[ Z e^{-i(\xi_k - i\gamma) t} + \ldots \right] \]

Consider the Fourier transform

\[ g(\vec{k}, \omega) = \int dt \, e^{i\omega t} g(\vec{k}, t) \]

\[ = \frac{Z}{\omega - \xi_k + i\gamma} + \ldots \]
$g(k,t)$ and $g(k,t)$ are examples of Greens functions. We see that the Greens function has a pole at the energy of the quasiparticle. The pole approaches the real axis as the quasiparticle lifetime increases (which happens as the excitation energy is lowered).

The overlap $Z$ appears as the residue associated with this pole. In general any system with well-defined quasiparticle excitations will have a "quasiparticle pole" and a non-zero "quasiparticle residue".

In practice it is very convenient to work with so-called extended and advanced Greens functions defined by

$$G_{x}(x_{1}, t_{1}, x_{2}, t_{2}) = \frac{-i}{\Sigma_{c} G_{c}} \text{Tr} \left( e^{-\beta (\hat{H} - \mu \hat{N})} \int \psi(x_{1}, t_{1}) \psi^{+}(x_{2}, t_{2}) \right)$$

$$G_{A}(x_{1}, t_{1}, x_{2}, t_{2}) = \frac{1}{\Sigma_{c} G_{c}} \text{Tr} \left( e^{-\beta (\hat{H} - \mu \hat{N})} \int \psi(x_{1}, t_{1}) \psi^{+}(x_{2}, t_{2}) \right)$$

(For generality we have defined these at $T \neq 0$)

$Z_{c}$ = grand partition function

Here time evolution is through the operator $e^{-i(\hat{H} - \mu \hat{N})t}$
This can be simply written as

$$G_R(x_1, t_1, x_2, t_2) = -i \langle \left\{ \psi(x), \psi^+(x) \right\} \rangle \delta(t_1 - t_2)$$

with the $\langle \ldots \rangle$ referring to both quantum and thermal averaging (and likely for the advanced Greens function).

It is instructive to obtain an exact expression for these in terms of the exact eigenstates and matrix elements of $\hat{H}$.

First (and for a translation invariant system) it is convenient to go to momentum space

$$G_R(k, t) = -i \langle \left\{ \psi(k), \psi^+(0) \right\} \rangle \delta(t)$$

$$= \int d^d x \ e^{-i (k - \mathbf{p}) \cdot x} G_R(x, t)$$

(Note that $G_R$ can only depend on $x_1 - x_2$, $t_1 - t_2$).

Insert a complete set of states

$$G_R(k, t) = -i \sum_n e^{-\beta E_n} \delta(n, m) \langle n | \psi(k, t) | m \rangle \langle m | \psi^+(0) | n \rangle$$

(with $E_n = \text{eigenenergy of } \hat{H} - \mu n$

$$= E_n - \mu n$)

(Note: The $1^{st}$ term is zero unless $N_m = N_{n+1}$ and

The $2^{nd}$ term is zero unless $N_m = N_{n-1}$).
\[ G_R(\mathbf{k}^t) = -\frac{i}{Z_G} \sum_{nm} e^{-\beta (\tilde{E}_n - \tilde{E}_m)} \left( e^{-i\tilde{E}_{mn}t} \left| \left< m \mid \psi^+_k \right| n \right|^2 + e^{-i\tilde{E}_{mn}t} \left| \left< m \mid \psi^+_k \right| n \right|^2 \right) \]

with \( \tilde{E}_{mn} = \tilde{E}_n - \tilde{E}_m \)

\[ \Rightarrow G_R(\mathbf{k}^t) = -\frac{i}{Z_G} \sum_{nm} e^{-\beta (\tilde{E}_n - \tilde{E}_m)} \left( e^{-i\tilde{E}_{mn}t} \left| \left< m \mid \psi^+_k \right| n \right|^2 + e^{-i\tilde{E}_{mn}t} \left| \left< m \mid \psi^+_k \right| n \right|^2 \right) \]

Now consider the Fourier transform

\[ G_R(\mathbf{k} \omega) = \frac{1}{(2\pi)^3} \int_0^\infty dt \ G_R(\mathbf{k}^t) e^{i\omega t} e^{-\gamma t} \quad \text{with} \quad \gamma = \omega^+ \]

The \( e^{-\gamma t} \) is to ensure convergence in the large-\( t \) limit of integration.

Then \( G_R(\mathbf{k} \omega) = \frac{1}{Z_G} \sum_{nm} e^{-\beta (\tilde{E}_n - \tilde{E}_m)} \left[ \frac{\left| \left< m \mid \psi^+_k \right| n \right|^2}{\omega - \tilde{E}_{mn} + i\eta} \right] \]

\[ + \frac{\left| \left< m \mid \psi^+_k \right| n \right|^2}{\omega + \tilde{E}_{mn} + i\eta} \]

In 1st term \( N_m = N_n + 1 \) \Rightarrow \( \tilde{E}_{mn} = E_m - E_n - \mu (N_m - N_n) \)

\[ = E_m - E_n - \mu \]

But in 2nd term \( N_m = N_n - 1 \) \Rightarrow \( \tilde{E}_{mn} = E_m - E_n + \mu \).
\[ G_R (\vec{k}, \omega) = \sum_{nm} e^{-\beta (E_n - \mu + \omega)} \left[ \frac{\langle m | \psi_k^+ | n \rangle}{\omega - (E_m - E_n + \mu) + i\eta} \right]^2 + \frac{\langle m | \psi_k^+ | n \rangle}{\omega + (E_m - E_n + \mu) + i\eta} \left[ \frac{\langle m | \psi_k | n \rangle}{\omega + (E_m - E_n + \mu) + i\eta} \right]^2 \]

Formally if we regard \( \omega \) as a complex number

\( G_R (\vec{k}, \omega) \) is analytic in the upper half-plane.

Reiterating

Using a similar "spectral representation" we can prove that \( G_A (\vec{k}, \omega) \) is analytic in the lower half-plane.

Can clearly write

\[ G_R (\vec{k}, \omega) = \int d\Omega \frac{A(\vec{k}, \Omega)}{\omega - \Omega + i\eta} \]

with

\[ A(\vec{k}, \Omega) = \frac{1}{Z_g} \sum_{nm} e^{-\beta E_n} \left[ \frac{\langle m | \psi_k^+ | n \rangle^2}{\omega - (E_m - E_n + \mu) + i\eta} \right]^2 + \frac{\langle m | \psi_k^+ | n \rangle^2}{\omega + (E_m - E_n + \mu) + i\eta} \left[ \frac{\langle m | \psi_k | n \rangle^2}{\omega + (E_m - E_n + \mu) + i\eta} \right]^2 \]

\( A(\vec{k}, \Omega) \) is known as the "electron spectral function"

and has all the important information about the spectrum of one-particle excitations.
Clearly \( A(\mathbf{k}, \omega) > 0 \).

Similarly we can write

\[
G_A(\mathbf{k}, \omega) = \int d\Omega \frac{A(\mathbf{k}, \Omega)}{\omega - \Omega - i\eta}
\]

Thus \( G_A(\mathbf{k}, \omega) = (G_R(\mathbf{k}, \omega))^* \) for \( \omega \) real.

Consider \( \text{Im} \ G_R(\mathbf{k}, \omega) = -\pi \int d\Omega \ A(\mathbf{k}, \Omega) \ S(\omega - \Omega) \)

\[
\Rightarrow \quad \boxed{A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} \ G_R(\mathbf{k}, \omega)}
\]

(Similarly \( A(\mathbf{k}, \omega) = \frac{1}{\pi} \text{Im} \ G_A(\mathbf{k}, \omega) \)).

Finally consider \( \int d\Omega \ A(\mathbf{k}, \Omega) \)

\[
= \frac{1}{Z_G} \sum_{nm} e^{-\beta E_n} \left[ |\langle m| \psi_k^+ |n \rangle|^2 + |\langle m| \psi_k^+ |n \rangle|^2 \right]
\]

\[
= \frac{1}{Z_G} \sum_n e^{-\beta E_n} \left[ \langle n | \psi_k \psi_k^+ + \psi_k^+ \psi_k | n \rangle \right]
\]

\[
= 1.
\]
A particularly useful form for $A(k,\omega)$ is obtained by interchanging the $(n,m)$ indices in the 1st term which then becomes

$$\sum_{nm} e^{-\beta \tilde{E}_m} \left| \langle n | \psi^+_k | m \rangle \right|^2 S(\omega + \tilde{E}_{mn})$$

$$= \sum_{nm} e^{-\beta (\tilde{E}_m - \tilde{E}_n)} e^{-\beta \tilde{E}_n} \left| \langle m | \psi_k | n \rangle \right|^2 S(\omega + \tilde{E}_{mn})$$

$$= \sum_{nm} e^{-\beta \tilde{E}_n} e^{\beta \tilde{E}_m} \left| \langle m | \psi_k | n \rangle \right|^2 S(\omega + \tilde{E}_{mn})$$

$$\therefore A(k,\omega) = \frac{1}{Z_0} \sum_{nm} e^{-\beta \tilde{E}_n} \left| \langle m | \psi_k | n \rangle \right|^2 S(\omega + \tilde{E}_{mn})$$

$$= \frac{1}{Z_0} \sum_{nm} e^{-\beta \tilde{E}_n} \left| \langle m | \psi_k | n \rangle \right|^2 S(\omega - (\tilde{E}_n - \tilde{E}_m))$$

with $f(\omega) = \frac{1}{e^{\beta \omega} + 1} = \text{Fermi function}$. The RHS is directly measured by "angle-resolved photoemission" experiments (ARPES). In ARPES an energetic high photon energy photon incident on the sample knocks out an electron. The energy and momentum of the ejected electron is measured.
Treating the electron ejection process within a "sudden approximation" the probability rate of ejecting electrons with a momentum and energy transfer \((\mathbf{p}, \omega)\) to the sample is given by Fermi's golden rule to be precisely \(A(\mathbf{p}, \omega) f(\omega)\).

The equilibrium state momentum distribution \(\eta_k = \langle \psi_k^+ \psi_k \rangle\) can also be obtained from \(A(\mathbf{p}, \omega)\).

Consider \(\int d\omega A(\mathbf{p}, \omega) f(\omega) = \frac{1}{Z} \sum_{nm} e^{-\beta E_n} |\langle m | \psi_k | n \rangle|^2\)

\[= \langle \psi_k^+ \psi_k \rangle\]

Now consider the form of the retarded Greens function at \(T=0\) for a Landau Fermi liquid with sharp quasiparticles. At \(T=0\)

\[G_R (\mathbf{p}, \omega) = \sum_m \left[ \frac{|\langle m | \psi_k^+ | g\rangle|^2}{\omega - (E_m - E_\omega - \mu) + i\eta} + \frac{|\langle m | \psi_k | g\rangle|^2}{\omega + (E_m - E_\omega + \mu) + i\eta} \right] \]
Following our earlier discussion, the 1st term corresponds to adding an electron to the ground state and will take the form
\[
Z \frac{e(\varepsilon_k - \mu)}{\omega - (\varepsilon_k - \mu) + i\gamma}
\]
with \( Z \) the quasiparticle weight and \( \gamma \) the decay rate.

Similarly, the 2nd term corresponds to creating a "quasi-hole" below the Fermi surface. Expect that the energy \( E_m - E_0 + \mu = \mu - \varepsilon_k \) so that this term will take the form
\[
Z \frac{e(\mu - \varepsilon_k)}{\omega - (\varepsilon_k - \mu) + i\gamma}
\]

\[
\therefore \text{ The total } G_k(k, \omega) \approx Z \frac{e(\mu - \varepsilon_k)}{\omega - (\varepsilon_k - \mu) + i\gamma} + \ldots
\]

The \( \ldots \) correspond to multi quasiparticle/hole states that are created by addition/removal of electrons.

Thus we see that \( G_k \) has a pole in the lower half-plane but is analytic in the upper half-plane as required.

The electron spectral function \( A(k, \omega) = -\frac{1}{\pi i} \text{Im } G_k(k, \omega) \)
\[
= \frac{1}{\pi i} \frac{\varepsilon_k^2}{(\omega - (\varepsilon_k - \mu) + i\gamma)^2} + A_{\text{inc}}(k, \omega)
\]
This is a Lorentzian at frequency $\omega \approx \omega_0$ peak $\approx \omega_0 - \mu$ with a width $\Gamma$. (A inc is the "incoherent" part.)

As $\Gamma \ll |\omega_0 - \mu|$ for small $|\omega_0 - \mu|$, the peak is sharply defined.

As $|\omega_0 - \mu| \to 0$, i.e., as we approach the Fermi surface, the Lorentzian becomes a S-function.

$$A(\mathbf{k}, \omega) \to Z \delta(\omega - (\omega_0 - \mu)) + A_{inc}.$$ 

$$= Z \delta(\omega - \nu_f k_f) + A_{inc}$$

where $\nu_f$ = Fermi velocity and $k_f$ = momentum parallel to Fermi momentum. $k_f = |\mathbf{k}| = |\mathbf{k}_F| = \text{deviation of momentum from Fermi surface.}$

Seeing such a quasiparticle peak in a photoemission experiment is a direct proof of the existence of quasiparticles.

At $T_f$ to the quasiparticle peak will be broadened into a Lorentzian with width $\Gamma \approx T_f^2$ even right at the Fermi surface.
For a non-interacting system

\[ A(k, \omega) = S(\omega - (E_k - \mu)) \]

i.e. \( Z = 1 \), \( T = 0 \), and \( \Delta n = 0 \).

Then the momentum distribution

\[ n(k) = \int_{-\infty}^{\infty} d\omega A(k, \omega) = \Theta(\mu - E_k) \text{ which is} \]

the familiar result. This has a jump \( \Delta n = 1 \) at the Fermi surface.

What about the interacting Fermi liquid?

Using \( A(k, \omega) \approx Z S(\omega - (E_k - \mu)) + \Delta n \mu \cos \theta \)

we see that at \( T = 0 \)

\[ n(k) = Z \Theta(\mu - E_k) + \text{background} \]

⇒ the true momentum distribution still has a jump discontinuity at the Fermi surface but the jump is

\[ \Delta n = Z < 1 \].

\[ \text{Diagram:} \]

\[ \text{Graph}: \]

\[ \text{Arrow}: \]
So long as \( Z \neq 0 \) Fermi liquid theory is an asymptotically correct description of the low energy excitations associated with the Fermi surface — in particular the predictions for low-T specific heat, susceptibility or transport will be correct.

However in a strongly correlated Fermi liquid metal \( Z \) may be very small so that it may be hard to resolve the \( \nu = 1 \) peak or the jump in the momentum distribution.

(E.g.: In heavy electron metals, theory predicts very small \( \nu \approx \frac{m}{M} \approx 10^{-2} \) or \( 10^{-3} \) but Fermi liquid theory still works very well at low-T)

Comments on ARPES: As we saw \( A(\vec{p}, \omega) f(\omega) \) can be directly measured by ARPES. This means that in principle the shape and size of the Fermi surface, the dispersion & lifetime of the quasiparticles can all be mapped out. Modern ARPES experiments have energy resolution of few meV (and angle resolution of 0.1-0.3 degrees). Also requirements in temperature and purity not as stringent as for quantum oscillations.
Fermions on a lattice: the Hubbard model.

A useful framework for thinking about the properties of
fermions with short-ranged interactions on a lattice is
provided by the Hubbard model

\[ H = -t \sum_{\langle i,j \rangle} \left( c_i^\dagger c_j + h.c. \right) + U \sum_i n_i (n_i - 1) \]

(Compare with Bose Hubbard model).

\( U > 0 \) \( \rightarrow \) on-site repulsion.

Generic filling, not too large \( U \): expect particles will be
mobile & will form a correlated Fermi liquid.

Guess for \( gd \) state wave function:

Slater-Jastrow \( |\psi_{\text{gd}}\rangle \propto e^{-\sum_{ij} n_i n_j} |\psi_{\text{free Fermi gas}}\rangle \)

Simple example: Assume \( \epsilon_{ij} \neq 0 \) only for \( i = j \)

This gives the Gutzwiller wave function

\[ |\psi_{\text{Gutz}}\rangle \propto e^{-\sum_i n_i^2} |\psi_{\text{free}}\rangle \]

Jastrow factor \( e^{-\sum n_i \eta_i^2} \propto e^{-\sum u (\eta_i^2 + \eta_j^2 + 2n_i n_j)} \)
\[ e^{a} = \sum_{n} \frac{(-1)^{n}}{n!} \left( \frac{a}{n!} \right)^{n-1} \]

As \( n_{\uparrow} n_{\downarrow} = 0 \) or 1, can rewrite as

\[ \prod_{i} \left( 1 - (1 - g) n_{\uparrow} n_{\downarrow} \right) = g^{\frac{1}{2}} n_{\uparrow} n_{\downarrow} \]

\[ (g = e^{-2u}) \]

\[ 1^{|\text{gate}|} \propto g^{D} |\psi_{\text{free}}\rangle \]

where \( D = \text{total \# of doubly occupied sites} \).

The \( g^{D} \) factor suppresses amplitude in \( g\text{d} \) state wavefuncition configurations with large \# of doubly occupied sites.

For the special case where \( \langle n_{\uparrow} \rangle = 1 \), if \( U \) is large,

can again get Mott insulating behavior (just as for bosons).

But fermions, in this large-\( U \) limit, must solve we are

confronted with the issue of what to do with the spin

degree of freedom which remains active at scales \( \ll U \) in the

Mott insulator.

Indeed at 0th order of pert. theory in \( 1/U \), there are \( 2^{N} \)

degenerate \( g\text{d} \) states corresponding to the various possible spin states.
This degeneracy will be split at higher orders of $\xi$.

Solve by deriving effective Hamiltonian that lives in spin space.

At $O(\xi^0)$, $H_{\text{eff}} = J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j$

with $J = 4\xi^2/\mu > 0$.

$J > 0$ as then nearest neighbor sites have spins anti-aligned allowing for a virtual hop that decreases K.E.

At higher orders, will get more complicated terms.

The fate of spin models such as the above is the problem of "quantum magnetism".

Most common fate: AF ordering of local moments but more interesting behavior is possible if lattice is "frustrated" or if $J/U$ is not too small so that higher order terms in $\xi$ must be kept in $H_{\text{eff}}$.

(Will discuss some of these latter.)

A central question:

How does the Fermi liquid phase evolve into the Mott insulator?
For conciseness, consider a non-bipartite lattice at \( \frac{1}{2} \) filling, i.e., \( \langle n_i \rangle = 1 \).

E.g., \( \square \) or \( \triangle \) lattices.

Why non-bipartite? In the bipartite case, typically non-interactivly

FS is perfectly nested, i.e., has flat portion

E.g.: Square lattice

This enhances \( (T, H) \) charge/spin susceptibility (directed)

\( \Rightarrow \) arbitrary weak U opens a gap & makes the system an insulator.

In a non-bipartite lattice, metallic phase stable up to \( \frac{U}{\mu} \) .

Mott

Insulator

Fermi

Liquid

A similar question can also be asked when the Mott insulator

is "depicted" by tuning a chemical potential.

\( \frac{U}{\mu} \)

\( \mu \)

At large enough doping, expect

\( \mu \) recovery FL.

Fermi

Liquid

\( \mu \)