1 Recap

1.1 Determinant

1.1.1 Algebraic View

The determinant of a square matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det A = \sum_{\sigma} \left( (-1)^{\text{sign}(\sigma)} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)} \right)$$

where $\sigma$ iterates over any permutation of $\{1, ..., n\}$ and $\text{sign}(\sigma)$ is the parity of $\sigma$.

An equivalent definition (more computationally convenient) for the determinant is

$$\det A = (\pm 1) \cdot (\text{product of pivots in ref}(A)),$$

where the sign depends on the parity of the number of row exchanges in the REF.

1.1.2 Geometric View

The determinant of a square matrix $A \in \mathbb{R}^{n\times n}$ is the scaling factor between $\text{vol}(S)$ and $\text{vol}(\phi_A(S))$ taking into account handedness; where $\phi_A(S) = \{Ax : x \in S\}$ is the region which $\phi_A$ maps $S$ into.

1.1.3 Properties

Let $A, B$ be $n \times n$ matrices. Let $C$ be another square matrix and $D$ be a matrix with proper dimension.

1. Swapping two rows (or two columns) of $A$ negates the determinant.
2. Adding a row to another row does not change the determinant.
3. Adding a column to another column does not change the determinant.
4. Multiplying a row/column by a scalar $c$ changes the determinant by a factor of $c$.
5. $\det A^T = \det A$
6. $\det AB = \det A \cdot \det B$
7. $\det(A + B) \neq \det A + \det B$
8. $\det \begin{bmatrix} A & D \\ 0 & C \end{bmatrix} = \det A \cdot \det C.$
1.2 Square Matrices Revisited

Let \( A \) be an \( n \times n \) square matrix. Then, the following statements are equivalent.

1. \( A \) is invertible, i.e. \( A^{-1} \) exists
2. \( A \) has both a left inverse and a right inverse
3. The columns of \( A \) are linearly independent
4. The rows of \( A \) are linearly independent
5. \( Ax = b \) is uniquely solvable for every \( b \in \mathbb{R}^n \)
6. \( N(A) = \{0\} \)
7. \( C(A) = \mathbb{R}^n \)
8. \( \text{Rank}(A) = n \)
9. \( \det A \neq 0 \)

1.3 Projection

Suppose that we want to find the orthogonal projection of a given vector \( w \) onto a \( k \)-dimensional subspace \( V \). In other words, we want to find a vector \( v = \text{proj}_V w \) for which

1) \( v \in V \), and 2) \( (w - v) \perp u \) for all \( u \in V \).

Suppose that \( \{v_1, \ldots, v_k\} \) is an orthonormal basis of \( V \). Then,

\[
v = \text{proj}_V w = \sum_{i=1}^{k} (w \cdot v_i) v_i = \sum_{i=1}^{k} v_i \cdot (v_i \cdot w) = \sum_{i=1}^{k} v_i \cdot (v_i^T w)
\]

\[
= \sum_{i=1}^{k} (v_i v_i^T) w = \left( \sum_{i=1}^{k} v_i v_i^T \right) w = Pw
\]

when \( P = \sum_{i=1}^{k} v_i v_i^T \) is an \( n \times n \) matrix. The matrix \( P \) is the orthogonal projection matrix onto the subspace \( V \), and can also be written as

\[
P = V V^T, \quad \text{where} \quad V = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \in \mathbb{R}^{n \times k}.
\]

Note that \( P \) is symmetric (i.e. \( P^T = P \)) and \( \text{rank}(P) = \text{rank}(V) = k \).

Another interesting property of \( V \) is that \( V^T V = I_k \). Indeed, this is a necessary and sufficient condition for \( \{v_1, \ldots, v_k\} \) to be orthonormal.

2 Exercises

1. Use determinant properties to show that if \( A \) and \( B \) are square matrices such that \( AB \) is invertible, then both \( A \) and \( B \) are invertible.

2. Let \( A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & \alpha & -5 \\ -1 & -1 & 2 \end{bmatrix} \). What values of \( \alpha \) makes \( A \) not invertible?
3. Consider the two subspaces $U$ and $W$ as follows.

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : u_1 - u_2 = 0, u_1 - u_3 = 0 \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} : w_1 + w_2 + w_3 = 0 \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

The two subspaces are orthogonal complements (check!). It can be easily seen that the given set of generators bases are actually bases (why?).

(a) Find an orthonormal basis of $U$. Use it to derive the projection matrix $P_U$ which projects vectors onto the subspace $U$.

(b) Find an orthonormal basis of $W$. Use it to derive the projection matrix $P_W$ which projects vectors onto the subspace $W$.

(c) Notice that $P_U + P_W = I_3$. It turns out that this is not a coincidence. For any orthogonal complement subspaces $U, W \subseteq \mathbb{R}^n$, the sum of their corresponding projection matrices is exactly $I_n$. Can you explain why it is always the case?

Hint: Use orthogonal decomposition. What exactly is each component of the decomposition?

4. Recall in 2D plane, a mirror matrix $M$ is the matrix for which given a vector $x$, then $Mx$ is the mirror image of $x$ across a line $L$ that passes through the origin.

(a) What can we tell about $M^2$ using the following fact: if we reflect a vector $v$ across the line $L$ twice, we end up with the original vector $v$.

(b) Using the answer from part (a), find the possible values of $\det M$.

(c) Take any 2-dimensional region $S$ you like. What is the region that is produced from left-multiplying any $x \in S$ by $M$? In other words, what is the region $\phi_M(S) = \{Mx : x \in S\}$?

(d) What is the volume of $\phi_M(S)$? How is to compared to the volume of $S$? What can we tell about $\det M$?

(e) Recall from problem set 2 that if $L$ makes angle $\theta$ with the $x$-axis, then we have $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. Is the expression consistent with part $a$ and $d$?