Welcome to 18.061. In this recitation, we will review some key points from Lecture 1 and do a few practice problems together. The remaining exercises are left to you for reference. Solutions will be released before next recitation.

A friendly reminder to 1) sign up for Piazza which is our primary Q&A platform, and 2) install and try out Julia at your convenience. Do not worry if you’re new to the language. We will have a Julia session at some point.

1 Recap
We went over some basic definitions and concepts of linear algebra.

1.1 Vectors
- A **vector** is a tuple of numbers.
- The **dimension** of the vector is the size of the tuple.
- A vector has a **magnitude** and a **direction**.
- The magnitude of a $n$-dimensional real vector $u$, or $\|u\|$, is the scalar $\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$, where each $u_i$ is an entry of the vector $u$.
- Unit vectors have a magnitude of 1.

1.2 Vector/Scalar Operations
- We can multiply a vector by a scalar (scalar multiplication).
  \[
  \lambda \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}
  \] (1)
- We can add two vectors of the same dimension (vector addition).
  \[
  \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}
  \] (2)
- Another operation called the **inner product** takes two vectors of the same dimension and gives a scalar. In particular, given two $n$-dimensional vectors $u$ and $v$, their inner product, denoted by $u \cdot v$, is
  \[
  u \cdot v = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.
  \] (3)

Vectors can be used to represent data, e.g. velocity vectors, feature vectors, etc.
1.3 Matrices

We can also represent data (or operations on data) by using a matrix, which is a rectangular array of numbers. Vectors and matrices are closely related.

If the dimensions are compatible, we can take a matrix-vector product:

\[
\begin{bmatrix}
  a & b \\
  c & d \\
  e & f
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
=
\begin{bmatrix}
  ax + by \\
  cx + dy \\
  ex + fy
\end{bmatrix}
\]  

(4)

The result can also be interpreted as a linear combination of the columns of the matrix:

\[
x \begin{bmatrix}
  a \\
  c \\
  e
\end{bmatrix} + y \begin{bmatrix}
  b \\
  d \\
  f
\end{bmatrix}
\]  

(5)
2 Exercises

2.1 During recitation

1. Consider the expression

\[ a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \]

(a) Find \( \begin{bmatrix} m \\ n \end{bmatrix} \) when \( a = 2, b = 4 \).

(b) Find the inner product between the vectors \( x \) and \( y \).

(c) Re-write the equation above in matrix-vector product form.

(d) Find values for \( a \) and \( b \), such that \( m = 5, n = 10 \).

*Hint: write it as a system of linear equations.*

2. Let

\[
A = \begin{bmatrix}
-1 & 1 \\
2 & 3 \\
0 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 4 & 7 \\
-2 & -5 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Which of the following matrix products are defined?

(a) \( AB \)

(b) \( CA \)

(c) \( ACB \)

(d) \( CCCCCCCCCCC \)

3. True or False: A vector (with real coordinates) has magnitude 0 if and only if it is the zero-vector, i.e. it consists of only zeroes.

4. True or False: If a vector is multiplied by a scalar \( c \), its magnitude also changes by a factor of \( |c| \).

5. In this exercise, we compute matrices associated to certain geometric transformations of vectors:

(a) Find a \( 2 \times 2 \) matrix such that when you multiply a 2-dimensional vector by it, the result is the reflection of the vector across the origin.

(b) Find a \( 3 \times 3 \) matrix such that when you multiply a 3-dimensional vector by it, it swaps the second and third coordinates of the vector.
2.2 After recitation (optional)

6. Let

\[
A = \begin{bmatrix}
-1 & 1 \\
2 & 3 \\
0 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 4 & 7 \\
-2 & -5 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad D = \begin{bmatrix}
2 \\
-1 \\
0
\end{bmatrix}.
\]

Which of the following matrix/vector products are well defined?

(a) BA
(b) AC
(c) BC
(d) ABADDAD
(e) AAA
(f) ABC
(g) ABD
(h) BACD
(i) ACBD
(j) BACBACBACBACBAC

7. True or False: For any vector \( u \), we have \( \|u\|^2 = u \cdot u \).

8. True or False: Let \( a \) and \( b \) be vectors of the same dimension. Then \( a \cdot b = b \cdot a \) if and only if \( a = b \).

9. Consider the following matrix \( A \) and vector \( y \). Does there exist a vector \( x \) such that \( Ax = y \)?

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
0 & 3 & 4 \\
0 & 0 & 0
\end{bmatrix}, \quad y = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

10. If you have a \( 4 \times 4 \) matrix \( A \), which 4-dimensional vector \( x \) can you choose such that \( Ax \) is the second column of \( A \)?

11. If you have a 4-dimensional vector \( x \), which \( 4 \times 4 \) matrix \( A \) can you choose such that \( Ax \) has its first/second/third/fourth entry being identical/double/triple/quadruple of \( x \)'s.

12. Install Julia. Test it out by defining a few vectors and matrices, and trying to add/multiply them. Also try to add/multiply things of the wrong dimensions, and see what happens.

For instance, you can define a matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and a vector \( x = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \), and multiply them by typing the following:

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad A \times x
\]

More information on how to download and try Julia online can be found in Piazza.
3 Solutions

1. (a) \[
\begin{bmatrix}
m \\
n
\end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 + (-4) \\ 6 + 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix}
\]

(b) \[x \cdot y = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 1 \cdot (-1) + 2 \cdot 3 = 5.\]

(c) \[
\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}.
\]

(d) \[
\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ 3a \\ 2b \end{bmatrix} + \begin{bmatrix} -b \\ 3a + 2b \end{bmatrix}. \]

Therefore, we have \[a - b = 5 \text{ and } 3a + 2b = 10.\] These equations together solve to \[a = 4 \text{ and } b = -1.\]

2. Matrices/vectors products are defined if and only if inner dimensions match.

Defined: (a) \(AB\), (c) \(ACB\), (d) \(CCCCCCCCCCCCCCC\).

Undefined: (b) \(CA\).

3. True. If a vector \(v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\) has magnitude 0, then \(v_1^2 + v_2^2 + \cdots + v_n^2 = 0\) which occurs if and only if \(v_1 = v_2 = \cdots = v_n = 0.\)

4. True. Let’s say we have a vector \(v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\). Then \(cv = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}\). So we have

\[
\|cv\| = \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}
\]

\[
= \sqrt{c^2(v_1^2 + v_2^2 + \cdots + v_n^2)}
\]

\[
= |c| \cdot \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\]

\[
= |c| \cdot \|v\|.
\]

5. (a) A 2-dimensional vector \(\begin{bmatrix} x \\ y \end{bmatrix}\) has a reflection across the origin

\[
\begin{bmatrix}
-x \\
-y
\end{bmatrix} = x \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Therefore, the matrix representing this transformation is \(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\).

(b) A 3-dimensional vector \(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\) must yield a result \(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\) which is

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]
Therefore, the matrix representing this transformation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

6. Matrices/vectors products are defined if and only if inner dimensions match.
   Defined: (a) BA, (b) AC, (g) ABD, (i) ACBD, (j) BACBACBACBACBAC.
   Undefined: (c) BC, (d) ABADDAD, (e) AAA, (f) ABC, (h) BACD.

7. True. Both terms are identical since
   $$\|u\|^2 = \left(\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\right)^2 = u_1^2 + u_2^2 + \cdots + u_n^2$$
   $$u \cdot u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1^2 + u_2^2 + \cdots + u_n^2.$$  

8. False. The values of $a \cdot b = \sum_{i=1}^{n} a_i b_i$ and $b \cdot a = \sum_{i=1}^{n} b_i a_i$ are the same regardless of $a_i$’s and $b_i$’s.

9. No, there are no possible solutions because any vector multiplied by the last row of $A$ will always equal 0.

10. Let’s say $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then
    $$Ax = x_1 \cdot (A’s 1^{\text{st}} \text{ column}) + x_2 \cdot (A’s 2^{\text{nd}} \text{ column}) + x_3 \cdot (A’s 3^{\text{rd}} \text{ column}) + x_4 \cdot (A’s 4^{\text{th}} \text{ column}).$$

    Since we want $Ax$ to be just the second column of $A$, it follows that $x_1 = x_3 = x_4 = 0$

    and $x_2 = 1$. Therefore, $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

11. Let’s say $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then
    $$Ax = x_1 \cdot (A’s 1^{\text{st}} \text{ column}) + x_2 \cdot (A’s 2^{\text{nd}} \text{ column}) + x_3 \cdot (A’s 3^{\text{rd}} \text{ column}) + x_4 \cdot (A’s 4^{\text{th}} \text{ column}).$$

    On the other hand, we want
    $$Ax = \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \\ 4x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$
Therefore, the first/second/third/fourth columns are $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$ respectively. This means $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$