Recall: A <u>vector space</u> over field k is a set V with two operations:

(1) addition $+ : V \times V \to V$ abelian group, $0 \in V$

(2) scalar multiplication $\times : k \times V \to V$ associative, distributive, $1v = v$, $0v = 0$.

Def: Given $v_1, \dots, v_n \in V$,

- $\mathrm{span}(v_1, \dots, v_n) = \{a_1 v_1 + \dots + a_n v_n \mid a_i \in k\}$ smallest subspace of V containing $v_1, \dots, v_n$
- $v_1, \dots, v_n$ are <u>linearly independent</u> if $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$
- $(v_1, \dots, v_n)$ are a <u>basis</u> if they are linearly independent and span V.

($\Rightarrow$ any element of V can be expressed <u>uniquely</u> as $\sum a_i v_i$ for some $a_i \in k$)

- Say V is finite-dimensional if $\exists$ finite set that spans V.
- We've seen:
  - $\to$ if $\{v_1, \dots v_n\}$ spans V, can select a subset of $\{v_i\}$ that forms a basis.
  - $\to$ if $\{v_1 \dots v_n\}$ are linearly indept, can add elements to form a basis.
  - $\to$ any two bases of V have same # elements, called the <u>dimension</u> of V.

* Given a basis $(v_1, \dots, v_n)$ of V, we get a linear map $\varphi : k^n \to V$, $(a_1, \dots, a_n) \mapsto \sum a_i v_i$

Linear independence $\Longleftrightarrow$ φ injective

spanning V $\Longleftrightarrow$ φ surjective, so φ is an isomorphism!

Every finite-dim. vector space $/k$ is isomorphic to k^n for $n = \dim V$.

(+ basis gives a specific <u>choice</u> of such an isomorphism).

Def. Let V, W be vector spaces $/k$. A <u>homomorphism</u> of vector spaces, or <u>linear map</u>, $\varphi : V \to W$, is any map that is compatible with the operations:

$$\varphi(u+v) = \varphi(u) + \varphi(v), \quad \varphi(\lambda v) = \lambda \varphi(v) \quad \forall \lambda \in k, \ \forall u, v \in V.$$

$\mathrm{Hom}(V, W) = \{\text{linear maps } V \to W\}$ is a vector space.

* Given bases $(v_1 \dots v_n)$ of V and $(w_1 \dots w_m)$ of W, we can represent a linear map $\varphi \in \mathrm{Hom}(V, W)$ by an $m \times n$ <u>matrix</u> $A \in \mathcal{M}_{m,n}$. This amounts to:

$$(*) \quad \begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \text{basis } \simeq\uparrow & & \uparrow\simeq \text{ basis} \\ k^n & \xrightarrow{A} & k^m \end{array}$$

Write $A = (a_{ij})_{\substack{1 \le i \le m \text{ rows} \\ 1 \le j \le n \text{ columns}}} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

$A : k^n \to k^m$ by multiplication w/ column vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Notation: $A = \mathcal{M}(\varphi, (v), (w))$ <u>the matrix of φ</u> in given bases

* The entries of A are characterized by : $\varphi(v_j) = \sum_{i=1}^{m} a_{ij} w_i$.

Ie: the columns of A give the components of $\varphi(v_1), \dots, \varphi(v_n)$ in the basis $\{w_1, \dots, w_m\}$.

Representing any element $x \in V$ as $x = \sum_{i=1}^{n} x_i v_i \leftrightarrow$ column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
and similarly for $y = \varphi(x) \in W$, $y = \sum y_i w_i \leftrightarrow Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = AX$.

* As a memory aid, the isom. $k^n \xrightarrow{\sim} V$ given by the basis can be written symbolically as multiplication of row & column vectors $\underbrace{(v_1 \dots v_n)}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum x_i v_i$.

⚠ these aren't numbers !!

$\varphi\big((v_1 \dots v_n) X\big) = (w_1 \dots w_n) AX$. (compare (*) above)

* This construction gives an isomorphism between the vector spaces $\mathrm{Hom}(V,W)$ and $\mathcal{M}_{m,n}$! In particular $\dim \mathrm{Hom}(V,W) = \dim \mathcal{M}_{m,n} = mn$.
linear maps $\longleftrightarrow$ matrices

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* Change of basis: What if we choose different basis for V and/or W?

If we change basis from $(v_1 \dots v_n)$ to $(v'_1 \dots v'_n)$, write $v'_j = \sum_{i=1}^{n} p_{ij} v_i$
and get an $n \times n$ matrix P whose j^{th} column gives the components of v'_j in the basis $(v_1 \dots v_n)$. Symbolically $(v'_1 \dots v'_n) = (v_1 \dots v_n) P$.

So ; $(v'_1 \dots v'_n) X' = (v_1 \dots v_n) P X'$ ie. the element of V described by a column vector X' in new basis is described by $X = PX'$ in old basis.

More conceptually: $P = \mathcal{M}(id_V, (v'), (v))$!

Do the same for W, but proceed in inverse direction, let $Q = \mathcal{M}(id_W, (w), (w'))$
ie. $(w_1 \dots w_m) = (w'_1 \dots w'_m) Q$.

Hence: $\varphi\big((v'_1 \dots v'_n) X'\big) = \varphi\big((v_1 \dots v_n) P X'\big) = (w_1 \dots w_m) APX'$
$= (w'_1 \dots w'_m) QAPX'$

ie. $\mathcal{M}(\varphi, (v'), (w')) = QAP$.

* In particular, if $V = W$ and change basis, for $\varphi \in \mathrm{Hom}(V,V)$,
$A = \mathcal{M}(\varphi, (v), (v))$ and $A' = \mathcal{M}(\varphi, (v'), (v'))$ are related by $A' = P^{-1}AP$.

$\longrightarrow$ But... the whole point of linear algebra is to avoid all this and work with linear maps in a coordinate-free language as much as possible.

- Direct sums and products of vector spaces

Given vector spaces V and W, $V \oplus W = V \times W = \{(v,w) \mid v \in V, w \in W\}$
(with componentwise operations).

Similarly given n vector spaces, $V_1 \oplus \dots \oplus V_n = V_1 \times \dots \times V_n = \{(v_1, \dots v_n) \mid v_i \in V_i\}$

But for infinite collection $(V_i)_{i \in I}$, we have two different constructions:

$$\bigoplus_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i, \text{ only finitely many } v_i \neq 0\} \quad \text{vs.} \quad \prod_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i\}$$

Ex: $\bigoplus_{n \in \mathbb{N}} k \simeq k[x]$ vs. $\prod_{n \in \mathbb{N}} k \simeq k[[x]]$ formal power series.

returning to finite case...

- Sums and direct sums of subspaces:

Def: Given subspaces $W_1, \dots, W_n \subset V$ of some vector space V,
- the span of $W_1, \dots, W_n$ is $W_1 + \dots + W_n = \{w_1 + \dots + w_n \mid w_i \in W_i\} \subset V$.
  Say the W_i span V if $W_1 + \dots + W_n = V$.
- the W_i are independent if $w_1 + \dots + w_n = 0$, $w_i \in W_i \Rightarrow w_i = 0 \ \forall i$.
- if the W_i are independent and span V, say we have a direct sum decomposition $V = W_1 \oplus \dots \oplus W_n$.

* Relation to the previous notion: $\forall i$ we have an inclusion map $W_i \hookrightarrow V$.

These assemble into a linear map
$$\varphi : \bigoplus W_i \longrightarrow V$$
$$(w_1, \dots, w_n) \longmapsto \sum w_i .$$

$W_1 \dots W_n$ span V $\Leftrightarrow$ φ surjective, independent $\Leftrightarrow$ φ injective.

If both hold, then φ is an isomorphism $\bigoplus W_i \xrightarrow{\sim} V$ and we have a direct sum decomposition.

In this case $\dim(V) = \sum \dim(W_i)$
(get a basis of V by taking the union of bases of $W_1, \dots, W_n$).

* Case of two subspaces:
- W_1, W_2 are independent iff $W_1 \cap W_2 = \{0\}$.
- $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.
- $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = 0$ and $\dim W_1 + \dim W_2 = \dim W$.

$w_1 + w_2 = 0$ iff $w_1 = -w_2 \in W_1 \cap W_2$

we'll see this soon

* Also note: given a subspace $W \subset V$, there exists another subspace W' st. $W \oplus W' = V$. (W' is definitely not unique!). To find W': take a basis $\{w_1, \dots w_r\}$ of W, complete it to a basis $\{w_1, \dots, w_r, w_1' \dots w_s'\}$ of V, let $W' = \mathrm{span}(w_1', \dots w_s')$.

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* <u>Rank and the dimension formula:</u>

given finite-dim. vector spaces V and W, and a linear map $\varphi: V \to W$,

- $\mathrm{Ker}(\varphi) = \{v \in V \,/\, \varphi(v) = 0\} \subset V$
- $\mathrm{Im}(\varphi) = \{w \in W \,/\, \exists v \in V \text{ st. } \varphi(v) = w\} \subset W$ are subspaces of V & W.
- $\dim(\mathrm{Im}\,\varphi)$ is called the <u>rank of φ</u>

<u>Prop:</u> $\dim \ker(\varphi) + \dim \mathrm{Im}(\varphi) = \dim V$.

<u>Pf:</u> start by choosing a basis $\{u_1 \dots u_m\}$ for $\ker \varphi$, and complete it to a basis $\{u_1 \dots u_m, v_1 \dots v_r\}$ of V. We claim $(\varphi(v_1), \dots, \varphi(v_r))$ is a basis of $\mathrm{Im}(\varphi)$. Indeed:

- if $w = \varphi(v) \in \mathrm{Im}\,\varphi$, then write $v = \sum a_i u_i + \sum b_j v_j$
 and get $\varphi(v) = \sum b_j \varphi(v_j)$ so $\{\varphi(v_j)\}$ span $\mathrm{Im}(\varphi)$
- if $\sum c_j \varphi(v_j) = 0$ then $\varphi(\sum c_j v_j) = 0$, so $\sum c_j v_j \in \ker(\varphi)$
 ie. $\sum c_j v_j = \sum a_i u_i$ for some $a_i \in k$.
 But since $\{u_1 \dots u_m, v_1 \dots v_r\}$ are linearly indep, this forces all $c_j = 0$ (and $a_i = 0$). Hence $\varphi(v_j)$ are linearly indep^t.

So now: since $\{\underbrace{u_1 \dots u_m}_{m = \dim \ker \varphi}, \underbrace{v_1 \dots v_r}_{r = \dim \mathrm{Im}(\varphi) = \mathrm{rank}\,\varphi}\}$ basis of V, $m + r = \dim V$. □

((u_i) basis of $\ker \varphi$) (($\varphi(v_i)$) are a basis of $\mathrm{Im}\,\varphi$)

<u>Corollary 1:</u> Given a linear map $\varphi: V \to W$, there exist bases of V and W in which the matrix of φ has the form

$$\begin{array}{cc} & \overbrace{\quad}^{r} \\ {\scriptstyle r = \mathrm{rank}\,\varphi}\left\{\right. & \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

<u>Proof:</u> take basis of V which is $\{v_1, \dots, v_r, \overbrace{u_1, \dots u_m}^{\text{basis of } \ker \varphi}\}$ as above, and complete $\{\varphi(v_1) \dots \varphi(v_r)\}$ (basis of $\mathrm{Im}\,\varphi$) to a basis of W. □

<u>Corollary 2</u>: For $W_1, W_2 \subset W$ subspaces, $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

<u>Proof</u>: consider the map from $V = W_1 \oplus W_2$ to W,

$$\varphi(w_1, w_2) = w_1 + w_2.$$

Then $\operatorname{Im}(\varphi) = W_1 + W_2$, $\ker(\varphi) = \{(u,-u) \mid u \in W_1 \cap W_2\} \simeq W_1 \cap W_2$

so $\dim \ker \varphi + \dim \operatorname{Im} \varphi = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

$$= \dim(V) = \dim(W_1) + \dim(W_2). \qquad \square$$

Next time: quotient space, dual space.