Lecture 5: MHD Energy Principle and Instabilities

Here:

- Formulate theory of instabilities in Rayleigh-Ritz variational form
  → displacement, Lagrangian form
  → self-adjointness
  → Basic $\delta W(\xi^k) \to$ energy principle

- $\delta W = \frac{1}{2} \int d^3 x \left\{ \begin{array}{l}
\frac{\partial^2}{\partial t^2} \delta \xi + \mathbf{2} \mathbf{ \xi} - \mathbf{3} \\
+ \mathbf{3} \mathbf{ R}^2 \mathbf{ (2 \xi \cdot \mathbf{3})} + (2 \mathbf{ \xi} \cdot \mathbf{3}) \\
- (3 \mathbf{ \xi} \cdot \mathbf{ \xi}) \mathbf{ \Delta} \mathbf{ \xi} - (3 \mathbf{ \xi} \cdot \mathbf{3}) \mathbf{ \Delta} \mathbf{ \xi} \\
\end{array} \right\}$

$\mathbf{Q} = \mathbf{0} \times \mathbf{3} \times \mathbf{B}_0$

- Investigate basic physics of several instabilities of interchange variety.
Lecture 5: Energy Principle and Instabilities

5.) Least Action and the Energy Principle on MHD

\[ \text{Introduction} \]

- We now arrive at the MHD energy principle, which is a highlight of MHD, plasma physics, and classical physics, in general.

\[ \text{Energy Principle} \rightarrow \text{stability} \]

\[ \text{E.E. till now} \]

- 218B - waves, etc.

\[ \text{218.4 - trivial instabilities (e.g., 2-stream, bump-on-tail, J-driven, etc.)} \]

\[ \text{Realistic plasmas \{lab, astro\} \rightarrow free energy \left( \frac{\partial P}{\partial \Omega} \text{ etc.} \right)} \]

\[ \text{\uparrow \text{Instabilities with complex dynamics}} \]

\[ \text{\uparrow \{Rayleigh-Benard \rightarrow O.S. Interchanges \rightarrow K, DP (includes Rayleigh-Taylor) \}} \]

\[ \text{\{relaxation, turbulence, shocks, \}} \]

\[ \text{\{limits on performance (lab) \}} \]

\[ \text{\{restrictions on morphology (lab and astro) \}} \]
- brute force, frontal assault on instabilities often leads to heavy casualties...

- need a simple criterion, i.e. a necessary/sufficient criterion to identify and characterize instabilities

  ➔ Energy Principle

  Rayleigh–Ritz

  Energy Principle is very much in spirit of R-K Variational Principle ➔ no surprise as both based on self-adjointness of linear operators

  Ps. 101

- Proceed via:

  - sketch of Principle of Least Action for Ideal MHD ➔ Lagrangian formulation (Kulsrud 4.7)

  N.B. This underlies formulation in terms of displacement...

  - MHD eigenmode equation (generalized simple wave studies so far), second order W

  ➔ energy principle (Kulsrud 7.1, 7.2) (Kadomtsev Article)
- applications (various)

1) Principle of Least Action for MHD

- For ideal MHD, can immediately write

\[ L = \int d^3x \left[ \frac{\rho v^2}{2} \right] - W \]  

\[ W = \int d^3x \left( \frac{\rho}{\sigma-1} + \frac{\beta^2}{8\pi} + \rho \Phi \right) \]  

\[ S' = \int dt + L \rightarrow \text{action} \]

and can derive MHD equations by \( \delta L = 0 \)

- key point: how parametrize trajectory variations??

\[ S = \int dt + \int dx \left[ \frac{1}{2} (\frac{dy}{dt})^2 - \frac{1}{2} \left( 1 + \frac{y^2}{(\frac{dy}{dx})^2} \right)^{\frac{1}{2}} - 1 \right] \]

\[ \delta L = \delta L/\delta y \Rightarrow \frac{\partial L}{\partial y_t} \delta y_t + \frac{\partial L}{\partial y_x} \delta y_x \]  

etc...
The analogy with string suggests displacement is a natural way to formulate the least action for ideal MHD.

A natural link of MHD dynamics to particle dynamics

\[ \mathbf{r}(\mathbf{r}_0, t) \rightarrow \mathbf{r} + \mathbf{f}(\mathbf{r}_0, t) \]

At \( t_1 \), original blob has

\[ \mathbf{r} = \mathbf{r}_0 + \mathbf{f}(\mathbf{r}_0, t) \]

How relate \( \mathbf{f}(\mathbf{r}_0, t) \) to Eulerian velocity?\n
\( \mathbf{r}_0 + \mathbf{f}(\mathbf{r}_0, t) + \mathbf{f}(\mathbf{r}_0, t) \)\n
\[ \mathbf{v}(\mathbf{r}_0 + \mathbf{f}(\mathbf{r}_0, t), t) \]

\[ \frac{\partial}{\partial t} - \nabla \cdot (\mathbf{f}(\mathbf{r}_0, t)) \]

\( \mathbf{f}(\mathbf{r}_0, t) \) satisfies 3 nonlinear ODEs with \( \mathbf{f}(\mathbf{r}_0, 0) = 0 \) as c.i.c.
A theory of ode's assures solution exists.

Now, as in wave theory, we write all changes in NHO quantities in terms of displacements, i.e.

\[
\begin{align*}
\delta r &= -\bigtriangledown \cdot \left[ \rho (x, t) \delta \phi (x, t) \right] \\
\delta p &= -\bigtriangledown \cdot \rho \delta \phi (x, t) - \delta \phi (x, t) \cdot \delta \rho (x, t) \\
\delta \mathbf{b} &= \bigtriangledown \times \left( \delta \mathbf{E} (x, t) \times \mathbf{B} (x, t) \right)
\end{align*}
\]

and

\[
\delta \mathbf{v} (x, t) = \mathbf{v} (x, t) \cdot \bigtriangledown \delta \phi (x, t) - \delta \phi (x, t) \cdot \bigtriangledown \mathbf{v} (x, t) + \partial \frac{\delta \mathbf{E} (x, t)}{\partial t}
\]

so now, we can consider \( \delta S \)

\[
\delta S = \int_{t_1}^{t_2} dt \int d^3x \, \delta L
\]

\[
= \int_{t_1}^{t_2} dt \int d^3x \left( \frac{\delta \rho \mathbf{v}^2}{2} + \rho \mathbf{v} \cdot \delta \mathbf{v} - \frac{\delta \rho}{\gamma - 1} \frac{\delta \mathbf{b}}{4\pi} - \delta \rho \phi \right)
\]
plugging in $\delta$ quantities

$$\delta S = \int_{t_1}^{t_2} \int d^3 x \left\{ \nabla \cdot \left( -p \delta \vec{E} \right) + \vec{E} \cdot \nabla \times (\nabla \times \vec{B}) \right\} + \int_{t_1}^{t_2} \int d^3 x \left( \frac{\rho \vec{V} \cdot \nabla \vec{E} + \Delta \vec{E} \cdot \nabla \rho}{\gamma - 1} \right)$$

$$= \frac{\delta E_B}{4 \pi} \left( t_2 - t_1 \right)$$

$$+ \frac{\Delta \vec{E} \cdot \nabla \rho}{\gamma - 1} \left( t_2 - t_1 \right)$$

Now $\frac{\delta E}{\rho} \bigg|_{t_1}^{t_2} = 0$ and $\frac{\delta \vec{E}}{\rho \vec{V}} \bigg|_{t_1}^{t_2} = 0$ at the boundary.

So under a lot of conditions (with b.c.'s)

$$\delta S = \int_{t_1}^{t_2} \int d^3 x \left\{ \delta \vec{E} \cdot \left[ \frac{\rho \vec{V}^2}{2 \rho} - \frac{\nabla \cdot (\rho \vec{V} \vec{V})}{\rho \vec{V}} - \rho \frac{\nabla \vec{V}^2}{2 \rho} \right] \right\}$$

$$- \frac{\delta \vec{E}}{\rho \vec{V}} \bigg|_{t_1}^{t_2} + \Delta \vec{E} \cdot \nabla \rho \bigg|_{t_1}^{t_2}$$

$$+ \frac{\delta \vec{E} \cdot (\nabla \times \vec{B}) \times \vec{B}}{4 \pi}$$
\[ ds = -\int dt + \int d^3x \cdot \frac{\partial S}{\partial t} + \int d^3x \cdot \Delta \cdot \left[ \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v v) \ight] + \nabla \rho - \nabla \times B + \rho \nabla \phi \]

So \[ ds = 0 \quad \text{and} \quad d^3x = 0 \quad \Rightarrow \]

\[ \frac{\partial}{\partial t} \left( \rho v \right) + \nabla \cdot (\rho vv) = -\nabla \rho + \nabla \times B - \rho \nabla \phi \]

and \[ \frac{\partial}{\partial t} \rho \phi = -\nabla \cdot (\rho v) \quad \Rightarrow \]

\[ \rho \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = -\nabla \rho + \nabla \times B - \rho \nabla \phi \]

\[ \Rightarrow \text{equation of motion of ideal MHD emerges as Lagrange's Equation.} \]

Note: For case of \( v = 0 \) \( \Rightarrow \) equilibrium solution then \( ds' = 0 \) given:

\[ \frac{\partial \rho}{\partial t} = \nabla \times B - \rho \nabla \phi \]
Moral of this story:

- Can derive MHD equations from Principle of Least Action
- Displacement is a useful way to formulate ideal MHD dynamics

Now this brings us to:

(c.3) Energy Principle - Simple Form

Consider inhomogeneous static equilibrium/initial state with:

\[
\begin{align*}
\mathbf{V} \cdot \mathbf{B} &= 0 \\
\mathbf{D} \cdot \mathbf{B} &= 0 \\
\mathbf{D} \times \mathbf{B} &= \frac{4 \pi J_0}{C}
\end{align*}
\]

and no flow or \{self-gravity\}

Further assume \( \text{rigid wall bounds system} \) \([1,2]\)

\[
\mathbf{V} \cdot \hat{\mathbf{n}} \bigg|_{\text{wall}} = 0
\]

\[
\mathbf{B} \cdot \hat{\mathbf{n}} \bigg|_{\text{wall}} = 0
\]
and now... perturb system from equim by displacement

\[ \varepsilon(t) = \varepsilon_0 (t) \]

so if \( t = 0 \):

\[ \frac{\partial \varepsilon(r)}{\partial t} = \varepsilon_0''(r) \]

keep only linear terms (in \( \varepsilon \))

\[ r = r_0 + \varepsilon(r_0, t) \]

and \( r_0 \rightarrow r \) in argument of perturbed quantities

\[ \varepsilon_0''(r) \]

\[ \begin{bmatrix} P(t, r) = P_0 - V \cdot (P_0 \cdot \varepsilon(r)) \\ P(t, r) = P_0 - \varepsilon P_0 \cdot A \cdot \varepsilon - P_0 \cdot A \end{bmatrix} \]

\[ \begin{bmatrix} B(t, r) = B_0 + \nabla \times (\varepsilon \times B_0) \\ B(t, r) = B_0 + \nabla \times [\nabla \times (\varepsilon \times B_0)] \end{bmatrix} \]

putting it into equation of motion (linearized)
\[
\rho \frac{\partial^2 \mathcal{E}}{\partial t^2} = F(\mathcal{E})
\]

where:

\[
F(\mathcal{E}) = \frac{1}{4 \pi} \left[ \mathbf{J} \times \mathbf{E} - \mathbf{J} \times \mathbf{B}_0 \right] \times \mathbf{B}_0 \\
+ \mathbf{J}_0 \times \left[ \mathbf{E} \times \mathbf{B}_0 \right] - \mathbf{E} \cdot \mathbf{D} \cdot \mathbf{E} \\
+ \mathbf{D} \left[ \mathbf{E} \cdot \mathbf{D} \mathbf{E}_0 + \varepsilon \mathbf{D} \cdot \mathbf{E}_0 \right]
\]

with b.c. \( \mathbf{E} \cdot \mathbf{n} = 0 \) on surface
\( \mathbf{B} \cdot \mathbf{n} = 0 \) on surface

**Key Point:**

\( \Rightarrow F(\mathcal{E}) \) is self-adjoint

\[
\int d^3x \left[ \mathbf{J} \cdot F(\mathcal{E}) \right] = \int d^3x \left[ \mathbf{E} \cdot F(\mathcal{E}) \right]
\]
→ to prove: see Kulstad, PhlM, 6
(coming on PhlM Set III)

= consider the following (an indirect proof) . . . . 
legends remain involved . . .

→ can write total energy, to second order (on displacement), as:

\[ E = \int d^3x \left( \frac{\rho_0(x) (\nabla \varepsilon)^2}{2} + W(x, \varepsilon) \right) \]

\[ 2^{nd} \text{order bit of:} \]
\[ \int \left( \frac{\nabla \cdot \varepsilon}{8\pi} \right) d^3x \]

Now:

→ \[ W = W_0 + W_1(\varepsilon) + W_2(\varepsilon, \varepsilon) \]

\[ \text{first order} \quad \text{second order} \]

→ total energy is conserved, for any \( \varepsilon \)

with initial conditions \( \varepsilon_0, \dot{\varepsilon}_0 \)

provided \( \varepsilon \cdot \dot{\varepsilon} = \varepsilon \cdot \dot{\varepsilon} = 0 \) (b.c.)
Now, \( \frac{dE}{dt} = 0 \implies \)

\[
\frac{dE}{dt} = \int d^3x \ p_0 \left\{ \frac{2}{c^2} \cdot \frac{\partial E}{\partial t} \right\} + \mathcal{W}_1 \left( \frac{\partial E}{\partial t} \right) \\
+ \mathcal{W}_2 \left( \frac{\partial E}{\partial t} \right) + \mathcal{W}_2 \left( \frac{\partial E}{\partial t} \right) = 0
\]

and \( p_0 \ \frac{\partial^2 \xi}{\partial t^2} = F(\xi) \implies \)

\[
\frac{dE}{dt} = \int d^3x \left[ \frac{\partial E}{\partial t} \cdot F(\xi) \right] + \mathcal{W}_1 \left( \frac{\partial E}{\partial t} \right) \\
+ \mathcal{W}_2 \left( \frac{\partial E}{\partial t} \right) + \mathcal{W}_2 \left( \frac{\partial E}{\partial t} \right)
\]

but since \( \frac{dE}{dt} = 0 \) is always true, it is true at \( t=0 \), a particular time

Setting \( \xi_0 = \eta \implies \)

\[
\int d^3x \cdot F(\xi) + \mathcal{W}_1 (\eta) + \mathcal{W}_2 (\eta, \xi) \\
+ \mathcal{W}_2 (\xi, \eta) = 0
\]
Now, \( W_1(\eta) = 0 \) so
\[
\int d^3x \ (\eta \cdot F(\xi)) + \left[ W_2(\eta, \xi) + W_2(\xi, \eta) \right] = 0
\]
on more clearly \(
\int d^3x \ (\eta \cdot F(\xi)) = -\left[ W_2(\eta, \xi) + W_2(\xi, \eta) \right]
\)
so RHS symmetric under \( \eta \leftrightarrow \xi \) interchange.

\[
\int d^3x \ (\eta \cdot F(\xi)) = \int d^3x \ (\xi \cdot F(\eta))
\]
c and have proved self-adjointness.

\( \Rightarrow \) Finally, useful to note that if now \( \eta = \xi \)

\[
W_2(\xi, \xi) = -\frac{1}{2} \int d^3x \left[ \xi \cdot F(\xi) \right]
\]

- handy expression for \( W_2 \) in terms \( l = 1 \).
so now have shown that:

\[ F(\mathbf{E}) \quad \text{self-adjoint} \]

\[ W_2(\mathbf{E}), \quad \text{the potential energy of displacement } \mathbf{E} \]

\[ \text{can be expressed as:} \]

\[ W_2(\mathbf{E}) = -\frac{1}{2} \int d^3x \left[ \mathbf{E} \cdot \mathbf{F}(\mathbf{E}) \right] \]

From these we show several important results:

- Reality of \( \omega^2 \) and "exchange of stabilities"
- Due to structure of instability in ideal MHD
- Orthogonality of eigenfunctions
- Variational structure

\( \omega \) reality of \( \omega^2 \), "exchange of stabilities"

\[ \mathbf{E} = \hat{E} A \exp(-i \omega t) \]

\[ -i \omega \mathbf{E} = \mathbf{F}(\mathbf{E}) \quad (1) \]

\[ \mathbf{F}^* \mathbf{E}^* = \mathbf{F}(\mathbf{E}^*) \quad (2) \]

Explicitly real
\[
- a \left( \epsilon \times \epsilon^* \right) \epsilon^* \epsilon = \epsilon^* \cdot F(\epsilon) - \epsilon \cdot F(\epsilon^*)
\]

and integrating \(\Rightarrow\)

\[- a \int (\omega^2 - \omega^*) \, d^3 x \, (\epsilon^* \epsilon) = \int \left[ \epsilon^* \cdot F(\epsilon) - \epsilon \cdot F(\epsilon^*) \right] = 0, \text{ by self-adjoint property}\]

\[\Rightarrow \quad \epsilon^* \epsilon \text{ real} \Rightarrow (\omega^2)^* = \omega^2\]

\[\omega^2 \text{ is real}\]

\[\omega^2 > 0 \Rightarrow \text{stability}\]

\[\omega^2 < 0 \Rightarrow \text{instability, but purely growing} \quad \Rightarrow \text{no oscillation}\]

\[\text{Contrast to instabilities with which you should be familiar:}\]

\[\Rightarrow \text{bump-on-tail} \quad \omega = \omega_k^0 + i \gamma_k\]

\[\text{wave + inverse dispersion} \quad \gamma \sim \frac{\partial^2}{\partial y^2}\]

\[\text{carrier}\]
two stream

\[ \Theta = 1 - \frac{\omega_0^2 - \omega_0^2}{\omega^2 (\omega - i\nu)^2} \]

\( \omega^2 \) reactive counter-part of bump on tail can have \( \omega^2 \) real

\( \Rightarrow \) beam + dissipation \( \Rightarrow \) negative energy wave \\
\( \Rightarrow \) dissipation \( \Rightarrow \) growth

\( \omega = \omega_0 + i\gamma \)

\( \omega = \omega_0 + i\gamma \quad \gamma = C \frac{\partial \phi}{\partial y} - C \frac{\partial \phi}{\partial y} \)

wave + competition of dissipation and dispersion

\( \Rightarrow \) ideal Rayleigh-Benard Convection

\[ \omega^2 = -\frac{\kappa H}{\kappa^2 + \kappa^2} \frac{\partial \xi}{\partial z} \quad \xi > 0 \]

of these, ideal MHD instabilities similar in structure to convection and \( \omega^2 \) real cores of 2-stream and different in structure from the others
4. In ideal MHD, instability defines structure of eigenfunction, i.e. \( \hat{\Xi} = \Xi(r, \vartheta) \).

M.B. In ideal MHD, only scale in problem is system size \( \to \) boundaries. Contrast Sweet-Parker reconnection \( \Delta/L < 1 \), a case of resistive MHD.

\[ \text{proceeding} \Rightarrow \]

Since \( \omega^2 \) real, \( \omega^2 \) must pass thru \( \omega^2 = 0 \) as the system evolves from stable to unstable.

\[ \text{this evolution is called "exchange of stabilities"} \]

\[ \Rightarrow \text{marginal displacement solves } \mathcal{F}(\Xi) = 0 \]
N.B. = Solution of \( F(\xi) = 0 \) gives linear stability boundary in parameter space.

\[ \text{(c)} \text{ orthogonality} \]

Consider two solutions to \(-\rho_0 \omega^2 \xi = F(\xi)\)

\[ -\rho_0 \omega^2 \xi_1 = F(\xi_1) \times \xi_2 \]

\[ -\rho_0 \omega^2 \xi_2 = F(\xi_2) \times \xi_1 \]

\[ -(\omega_1^2 - \omega_2^2) \int d^3x \rho_0 \xi_1 \cdot \xi_2 = \int d^3x \left[ \xi_2 \cdot F(\xi_1) - \xi_1 \cdot F(\xi_2) \right] \]

\[ = 0, \text{ by self-adjointness} \]

\[ \omega_1^2 \neq \omega_2^2 \Rightarrow \int d^3x \rho_0 \xi_1 \cdot \xi_2 = 0 \]

\[ \Rightarrow \text{orthonormality with weighting function } \rho_0. \]

The point of all this is that now we can set up a variational quadratic form, a la' beloved Sturm-Liouville theory.
\[-\rho \omega^2 \varepsilon = E(\varepsilon)\]

and \(\circ \varepsilon \cdot = \Delta\)

\[
\omega^2 = -\frac{\int d^3 x \varepsilon \cdot E(\varepsilon)/2}{\int d^3 \varepsilon^{3/2}}
\]

\[
= W_2(\varepsilon)/\int d^3 \varepsilon^{3/2}
\]

\[\Rightarrow \text{with } k(\varepsilon) = \int d^3 x \rho \varepsilon^{3/2}, \text{ have}\]

\[
\begin{align*}
\omega^2 &= \frac{W_2(\varepsilon)}{k(\varepsilon)} \\
\text{[Variational, Quadratic form]}
\end{align*}
\]

and we know that, since all requirements satisfied, that

\[\text{any trial } \varepsilon \text{ plugged into } W_2(\varepsilon)/k(\varepsilon) \quad \text{yields } \omega^2(\varepsilon) > \omega^2\]

\[\Rightarrow \text{the true eigenvalue, i.e., variational result is always upper bound.}\]
So, we know that:

- If we can find a trial \( \mathbf{E} \) such that
  \[
  W_2(\mathbf{E}) < 0
  \]
  - then, configuration is surely unstable

This yields the desired necessary and sufficient condition for instability, namely that it be possible to find a \( \mathbf{E} \) such that

\[
W_2(\mathbf{E}) < 0.
\]

Hereafter, we write \( W_2(\mathbf{E}) = \delta W(\mathbf{E}) \), so the HTO Energy Principle is just:

\[
\text{instability iff } \int \text{ well behaved } \mathbf{E} \text{ s.t. } \delta W(\mathbf{E}) < 0
\]
N. B.

in physical terms, E. P. \( \Rightarrow \) instability if one can find a displacement which lowers the energy. Note linear instability \( \Rightarrow \) change to \( \mathcal{O}(E^2) \) considered.

\[
\delta W(\mathcal{E}) = -\frac{1}{2} \int d^3x \, \mathcal{E} \cdot \mathcal{F}(\mathcal{E})
\]

so, now must manipulate \( \delta W \) into physically useful form, i.e., recall

\[
\begin{align*}
\mathcal{F}(\mathcal{E}) &= \frac{1}{4\pi} \left\{ \nabla \times \left[ \nabla \times (\mathcal{E} \times B_0) \right] \right\} \times B_0 \\
&\quad + \mathcal{E} \times \mathcal{B}_0 - \mathbf{E} \\
&\quad + \mathcal{E} \times \nabla \times \mathcal{B}_0 \\
&\quad + \nabla \left[ \mathcal{E} \cdot \mathcal{E} + \mathcal{E} \cdot \mathcal{E} \right] \\
&\quad + \mathcal{E} \cdot (\mathcal{E} \times \mathcal{B}_0)
\end{align*}
\]

\[
= \mathbf{F}_0 + \mathbf{F}_0 + \mathbf{F}_0 + \mathbf{F}_0
\]

Remember here, all \( B_0, \mathcal{E}_0, \mathcal{E}_0 \) etc., in homogeneous, and \( \mathcal{E} \cdot \mathcal{E}_0 \) and \( B \cdot \mathcal{E}_0 \) on boundary.
- remains to manipulate \(-\int \frac{E \cdot F(E)}{2} \, d^3x\) into "ollum crong" form

- key is sign of \(dW\) so seek to extract quadratic terms, as unambiguous.

⇒ let the crook begin!

(1) \[ dW_0 = -\frac{1}{2} \int \frac{\varepsilon \cdot E_0(E)}{2} \, d^3x \]

\[ = -\frac{1}{2} \int d^3x \frac{\varepsilon \cdot \left( \nabla \times \left( \nabla \times E_0 \right) \right)}{4\pi} \times E_0 \]

\[ = \frac{1}{8\pi} \int d^3x \left( \nabla \times \left( \nabla \times E_0 \right) \right) \cdot \varepsilon \times E_0 \]

\[ = \frac{1}{8\pi} \int d^3x \nabla \cdot \left[ \nabla \times (\varepsilon \times E_0) \times (\varepsilon \times E_0) \right] \]

\[ + \frac{1}{8\pi} \int d^3x \left( \nabla \times (\varepsilon \times E_0) \right) \cdot \left( \nabla \times (\varepsilon \times E_0) \right) \]

if \( \varphi \equiv \nabla \times (\varepsilon \times E_0) = \varphi^\perp \), from induction

\[ dW_0 = \int d^3x \frac{\varphi^2}{8\pi} + \frac{1}{8\pi} \int dS \cdot (\nabla \times (\varepsilon \times E_0)) \times (\varepsilon \times E_0) \]
\[ \Delta W_0 = -\frac{1}{8\pi} \int_{\text{surface}} ds \left[ \hat{n} \cdot B_0 \times \phi - (\hat{n} \cdot \partial_0) B_0 \cdot \phi \right] \]

\[ \Delta W_0 = \int d^3x \frac{\phi_0^2}{\sqrt{\gamma}} \]

\[ \Delta W_0 = -\frac{1}{2} \int d^3x \ \hat{\boldsymbol{\varepsilon}} \cdot J_0 \times \left[ \partial_0 \times (\hat{\boldsymbol{\varepsilon}} \times B_0) \right] \]

\[ = -\frac{1}{2} \int d^3x \ \hat{\boldsymbol{\varepsilon}} \cdot (J_0 \times \hat{\boldsymbol{\varepsilon}}) \]

\[ = +\frac{1}{2} \int d^3x \ J_0 \cdot (\hat{\boldsymbol{\varepsilon}} \times \hat{\boldsymbol{\varepsilon}}) \]

\[ \Delta W_0 = -\frac{1}{2} \int d^3x \ \hat{\boldsymbol{\varepsilon}} \cdot \nabla \left[ \rho_0 \hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}} \rho_0 \right] \]

\[ \hat{\boldsymbol{\varepsilon}} \rho_0 \hat{\boldsymbol{\varepsilon}} = 0 \text{ on boundary} \]

\[ \Delta W_0 = \int d^3x \left[ \frac{\rho_0 (\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}})^2 + (\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}) \hat{\boldsymbol{\varepsilon}} \cdot \partial_0 \rho_0}{2} \right] \]

Note: "Last but not least..."
\[ \mathbf{W}_0 = -\int \frac{d^3 \mathbf{x}}{2} \mathbf{E} \cdot \nabla \times \mathbf{E} \cdot \nabla \times \mathbf{D} \cdot \nabla \cdot \mathbf{D} \]

\[ = -\frac{1}{2} \int d^3 \mathbf{x} \left( \mathbf{E} \cdot \nabla \phi \right) \nabla \cdot \left( \mathbf{E} \cdot \nabla \phi \right) \]

so putting the whole mess together

\[ \mathbf{W} = \frac{1}{2} \int d^3 \mathbf{x} \left( \frac{\mathbf{E}^2}{4\pi} + \mathbf{J}_0 \cdot \mathbf{E} \times \mathbf{D} \right) \]

\[ + \kappa \mathbf{A}(\mathbf{x}) \left( \nabla \cdot \mathbf{E} \right)^2 + \left( \mathbf{E} \cdot \nabla \mathbf{D}(\mathbf{x}) \right) \nabla \cdot \mathbf{E} - \mathbf{E} \cdot \nabla \mathbf{D}(\mathbf{x}) \cdot \nabla \cdot \mathbf{D}(\mathbf{x}) \]

\[ \mathbf{E} = \mathbf{D} \times (\mathbf{E} \times \mathbf{B}) \]

Note: general characteristics

- \( \mathbf{E} \to > 0 \) \( \implies \) field line bending \( \implies \) always stabilizing \( \mathbf{W} > 0 \)
- \( \mathbf{D} \to > 0 \) \( \implies \) compression

Free energy sources:

- \( \mathbf{J}_0(\mathbf{x}) \) on \( \mathbf{E} \) \( \implies \) density gradient
- \( \mathbf{J}_0(\mathbf{x}) \) on \( \mathbf{D} \) \( \implies \) current profile
- \( \mathbf{A}(\mathbf{x}) \) in \( \mathbf{E} \) \( \implies \) pressure gradient
- Gravity and \( \mathbf{B} \) in \( \mathbf{E} \)

\( \Phi \) can make \( \mathbf{W} < 0 \), for certain profiles

\( \Phi \to \) free energy sources for instability.
Note:

- $dw$ is imprecise

- $dw$ does not reveal much about growth rate but

- very useful for simple quick assessment of stability

- can elucidate complex problem

- problem in which infer non equilibrium not possible.
Further developments in theory remain, but better to consider some examples.

(iii) Convection and Interchange Instabilities

→ A Simple Application of the Energy Principle

Consider 4 related examples:

a) Convection and the Schwarzschild Criterion
b) Rayleigh-Taylor Instability
c) Interchange Instability
d) Interchange Without Gravity

(2) Schwarzschild Criterion and Convection

ci.e. Stellar atmosphere

\[ \rho g = \frac{d\rho}{dz} \]

\[ \rho \to 0, \quad z \uparrow \Rightarrow \frac{d\rho}{dz} < 0, \quad \frac{d\rho}{dz} < 0 \]

and

\[ \rho \rho - \gamma = \text{const.} \] (basic means of heat transport)

is basic idea of convection, consider a virtual displacement of a slug/ blob of gas upward

→ physical argument