Problem Set 2

1) Consider a sphere of radius $a$ and mass $m$ in an ideal (Euler) fluid, which is incompressible and of density $\rho$.
   a) Compute the flow field around the sphere, if it moves at $u = u_0 \hat{z}$. In an ideal flow, normal velocity vanishes at the surface of the sphere.
   b) Calculate the energy in the flow field. Discuss your result.
   c) Now calculate the force needed to accelerate the sphere at acceleration $a_0 \hat{z}$.
   d) What is the needed force, in the limit $m \to 0$. Explain your result.
   Hint: What is the “effective inertia” here?

2) An incompressible fluid rotates with $\Omega = \Omega(r) \hat{z}$. This system supports inertial waves — i.e. collective modes related to inertia.
   a) Compute the dispersion relation $\omega = \omega(k_r, k_z)$.
   b) For what profile does instability result? Can you explain this using simple physics? Hint: Consider angular momentum conservation and the interchange of two annular fluid elements.

3) Consider a sphere of radius $a$ in an incompressible viscous fluid. Take Reynolds number $Re \ll 1$.
   a) Calculate the force of drag opposing the motion of the sphere at velocity $u_0 \hat{z}$.
      N.B.: this is the famous Stokes drag calculation. You’ll need a reference — pick any. But keep your calculation as simple as possible.
   b) Now consider the low Reynolds number drag on a piece of rigid paper of dimensions $a,b$, which moves at $u_0 \hat{z}$. Show the drag scaling is the same for face-on and edge-on motion? Why? N.B.: This is an “estimates” problem. Keep it simple.
4) Consider a simple system with kinetic equation

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E_{\text{ext}}(x,t) \frac{\partial f}{\partial v} = c(f). \]

Take \( f_0 \) initially Maxwellian, formed by a very slow collisional process. \( c(f) \) is negligible on dynamical time scales. \( E_{\text{ext}}(x,t) \) varies slowly in time and space.

a) Compute the linear response \( \delta f \) to \( E_{\text{ext}} \). From this, derive an expression for the conductivity. What physics determines the conductivity on dynamical time scales. Best work in transform space!

b) Repeat part (a), with \( c(f) \), a simple krook operator. Describe the different regimes of the conductivity and their cross-over value.

c) Use your result to derive a mean field evolution equation for \( \langle f \rangle \) — the average distribution function, on \( t < \tau_{\text{coll}} \). Caution: \( \langle f \rangle \neq f_0 \). Discuss the physics of \( \langle f \rangle \) evolution.

d) Assuming \( E_{\text{ext}} \) is maintained, with space time dependence, what can be said about the end state of \( \langle f \rangle \) evolution. Hint: Consider \( \int d\nu \langle f \rangle^2 \).

e) Now consider the full evolution, when collisionality is weak but finite. Describe the evolution of \( \langle f \rangle \) on long time scales \( t > \tau_{\text{coll}} \), in the presence of \( E_{\text{ext}}(x,t) \).

5) Consider an elastic dumbbell of Stokesian particles in a fluid at temperature \( T \), with kinematic viscosity \( v \) and spring constant \( k \). Dumbbells have radius \( a \).

\[ l \]

a) What is the relaxation time (the Zimm time) of the dumbbell, response to a stretching perturbation? Neglect inertia, but give the condition for this.

b) What is the mean square length of the dumbbell? Describe how \( \langle l^2 \rangle \) might evolve if \( T \) is varied slowly.
c) Revisit a, b for the case where the dumbbell is replaced by an elastic shape, which remains Stokesian, and of negligible inertia.

N.B. This problem is based on simple models of polymers.